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HIMACHAL PRADESH UNIVERSITY JOURNAL

Contents

An elliptic curve cryptosystem over finite fields

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Abstract: Matrices and elliptic curves play an important role in cryptography to provide the confidentiality of the message, integrity of data and authentication to the communicating parties. We propose a cryptosystem using specific triangular matrices and elliptic curves over prime finite fields.

Keywords: Triangular matrices, Elliptic curves, Finite field, Cryptography.
Mathematics Subject Classification: 94A60, 11T71, 14G50, 68P25, 01A80.

1 INTRODUCTION

In the present era of information, all types of data travel over the insecure channels. Therefore, the security of data has become an important issue in the rapidly growing use of internet. Cryptographic techniques provide security to the data which is transmitted on the insecure channels. Finite fields are widely used in cryptography, see [6, 11]. Some public key cryptosystems are based on the techniques of number theory which provides high stability against attacks but they use a large key space, see [9, 12, 23, 24]. Such cryptosystems are not preferred where memory space is limited and computational power is required high. The elliptic curves provide the alternative to cope up with such practical problems. The elliptic curve cryptosystems occupy less memory space, much efficient in computations and fast in encrypting and decrypting process. Due to the complexity of these cryptosystems, hackers face more difficulty to crack. Therefore elliptic curve cryptosystems are highly preferred in practical.

Koblitz [5] introduces the elliptic curves for the use in cryptography and proposed an elliptic curve cryptosystem. Also, Miller [10] proposes an independent cryptosystem using elliptic curves. Various researchers have shown their interest for the use of elliptic curves in cryptography. They have further added that how these cryptosystems are useful in bandwidth savings, smart cards, wireless devices, faster implementations and increase high computational efficiency, see [24]. Consequently elliptic curves have attracted many researchers to contribute in the field of cryptography.

Non singular matrices are invertible. Therefore, such matrices have gain importance in cryptography. Hill [3, 4] uses matrices and linear transformations to develop cryptosystems. There are many cryptographic algorithms which are based on matrices, see [21]. Climent et al. [2] give a non linear elliptic curve cryptosystem based on matrices. Like matrices, there are some structures in the literature known as rhotrices. These structures are used in the field of cryptography to enhance the security of the existing cryptosystems, see [13-20].

The difficulty of solving discrete logarithmic problem provides the security to some cryptosystems. Elgamal cryptosystem is secure due to this difficulty, see [22]. Mahalanobis [8] discuss the Elgamal cryptosystem over circulant matrices. Amounas et al.[1] uses circulant matrices and elliptic curves for encryption and decryption process. Multiplication and squaring process of elements is fast in triangular matrices which is important in various cryptosystems. We develop a cryptosystem using triangular matrices and the elliptic curves over finite fields. We also show the encryption and decryption process with the help of an illustration.

2 ALGORITHM OF PROPOSED CRYPTOSYSTEM

Consider the lower triangular matrix

$$A=[a_{ij}]_m=\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ : & : & : & 0 & 0 \\ : & : & : & : & 0 \\ a_{m1} & a_{m2} & a_{m3} & a_{m4} & a_{mm} \end{bmatrix},$$

where $a_{ij} \epsilon \mathbb{F}_p$ and p is prime, in the proposed cryptosystem. This matrix should satisfy the following conditions:

(i) The lower triangular matrix A should have determinant '1'.

(ii) The matrix A should have each row-sum '1'.

(iii) The order m of matrix should be prime and it should be primitive modulo p.

(iv) The characteristic polynomial A_x of matrix A after division by $(x-1)$ gives irreducible polynomial.

2.1 Generation of data matrix of the message: The data matrix using elliptic curve over finite field $\mathbb{F}_p$ is obtained as follows:

(i) Initially choose an elliptic curve equation over finite field $\mathbb{F}_p$.

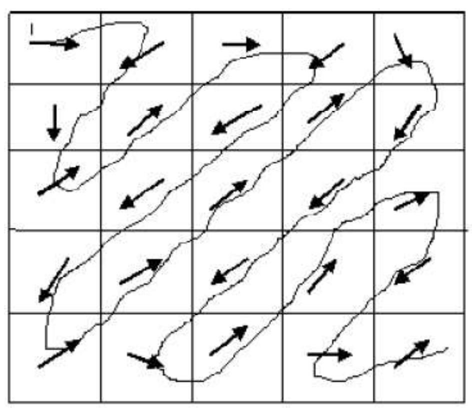
(ii) Obtain all the points of elliptic curve over finite field $\mathbb{F}$ 栩 .

(iii) Further, convert the elliptic curve points into binary form.

(iv) Now, construct a matrix B of order m .

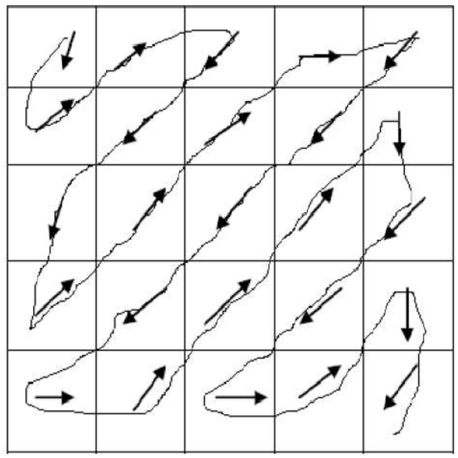
(v) Further, use the spiral traversal form (discussed below) over the matrix B and obtain the traversing data matrix C of the plaintext message M .

2.1.1 Spiral traversal form: In this form arrows shows that first entry of the first column goes to first entry of second column, first entry of second column goes to second entry of first column of the matrix, second entry of first column goes to third entry of first column, third entry of first column goes to second entry of second column. Likewise, all the entries are traversed.



Spiral traversal form for $m = 5$.

2.1.2 Reverse spiral traversal form: The reverse spiral traversal form is as follows. The first entry of first column goes to second entry of first column, second entry of first column goes to first entry of second column which further goes to first entry of third column. Likewise all the entries are traversed.



Reverse spiral traversal form for $m = 5$.

2.2 Encryption: To encrypt the message M sender does the following:

(i) He chooses a random integer r and publishes A^r.

(ii) Sender chooses one more integer randomly s (say) and keep it secret, then finds A^s and A^{rs}.

(iii) The ciphertext of the message M which is in the form of data matrix C, is thus obtained as $(T_1, T_2) = (k, CA^{rs})$, where k is the transpose of the first row of A^s.

(iv) Sender sends this cipher text to the receiver.

2.3 Decryption: To decrypt the message M receiver does the following:

(i) He extracts the first part k of the cipher text and form the lower triangular matrix with first row k which is same as the first row of A^s matrix.

(ii) Further, he finds A^{-rs} from the matrix obtained in the previous step and compute

$$(CA^{rs}).A^{-rs} = C.$$

(iii) He reverses all the operations which have been done during encryption process and finds the data matrix B.

(iv) Further, receiver converts the data sequence (digits) of matrix B to binary form such as $0 \to 00,\ 1 \to 01,\ 2 \to 10,\ 3 \to 11.$

(v) Now, receiver converts the sequence which is obtained in previous step to obtained elliptic curve points and then gets the original message back.

3 ILLUSTRATION OF PROPOSED ALGORITHM

Now we give illustration of the proposed algorithm to describe the process of encryption and decryption. Let us consider the message be **INDIA**.

3.1 Generation of data matrix of the message:

(i) Let us consider an elliptic curve equation

$$E_p \ : \ y^2 = (x^3 + 2x + 2) \bmod 17.$$

Using the Hasse theorem, see [pp. 174, 14] the order of elliptic curve is given by

$$|E_p| = 1 + p + \epsilon = 1 + p + \sum_{x \in \mathbb{F}_p} \frac{x^3 + ax + b}{p}$$
$$= 1 + p + \sum_{x \in \mathbb{F}_{17}} \frac{x^3 + 2x + 2}{17}.$$

Since $\mathbb{F}_p^*$ is a cyclic group. Therefore, by inspection we find $P = (5, 1)$ is a generator point on $E(\mathbb{F}_{17})$ and generate other points of $E(\mathbb{F}_{17})$ from this point. Doubling of the point P is as follows:

Let $P = (x_1, y_1) = (5, 1)$ be the point then the new point $2P = (x_2, y_{2)}$ is calculated as:

$$x_2 = \left(\frac{3x_1^2 + a}{2y_1}\right)^2 - 2y_1 = \left(\frac{3 \times 25 + 2}{2 \times 1}\right)^2 - 2 = 6,$$

$$y_2 = \left(\frac{3x_1^2 + a}{2y_1}\right)^2 (x_1 - x_2) - y_1 = 3.$$

Thus, we obtained the point (6, 3). Likewise, other points of are $E(\mathbb{F}_{17})$ as follows:

(7, 6), (3, 1), (0, 6), (9, 1), (5, 16) (6, 14), (7, 11), (9, 10) (10, 11), (10, 11), (13, 10), (16, 13), (13, 7), (0, 0), (10, 6) (16, 4), (0, 11), (3, 10).

The different alphabets used in the message **INDIA** are I, N, D, A. We shall randomly assign elliptic curve points to these four alphabets. For the remaining points, other alphabets and symbols are assigned to cover 19 points as follows:

EC points	Corresponding alphabets	EC points	Corresponding alphabet
(5, 1)	Y	(5, 16)	R
(6, 3)	S	(6, 14)	P
(7, 6)	A	(7, 11)	N
(3, 1)	B	(9, 10)	M
(0, 6)	T	(10, 11)	U
(9, 1)	E	(13, 10)	O
(10, 6)	C	(16, 13)	I
(16, 4)	( )	(13, 7)	D
(0, 11)	@	(0, 0)	Space
(3, 10)	#	-	-

Elliptic curve points and corresponding alphabets

Therefore, the elliptic curve points of the message are as follows:

I	(16, 13)
N	(7, 11)
D	(13, 7)
I	(16, 13)
A	(7, 6)

Now, conversion of elliptic curve points in binary form is as follows:

I	(16, 13)	(10000, 1101)
N	(7, 11)	(0111, 1011)
D	(13, 7)	(1101, 0111)
I	(16, 13)	(10000, 1101)

A	(7, 6)	(0111, 0110)

Padding zero to the extreme left in each tuple of the binary form points to obtain five bits tuples, we obtain

I	(16, 13)	(10000, 1101)	(10000, 01101)
N	(7, 11)	(0111, 1011)	(00111, 01011)
D	(13, 7)	(1101, 0111)	(01101, 00111)
I	(16, 13)	(10000, 1101)	(10000, 01101)
A	(7, 6)	(0111, 0110)	(00111, 00110)

The resulting padding tuples are clubbed in single string as follows:

I	(16, 13)	(10000, 1101)	(10000, 01101)	(1000001101)
N	(7, 11)	(0111, 1011)	(00111, 01011)	(0011101011)
D	(13, 7)	(1101, 0111)	(01101, 00111)	(0110100111)
I	(16, 13)	(10000, 1101)	(10000, 01101)	(1000001101)
A	(7, 6)	(0111, 0110)	(00111, 00110)	(0011100110)

The obtained string of 10 bits is then converted into decimal form by taking a sum of two, we obtain

I	(16, 13)	(10000, 1101)	(10000, 01101)	(1000001101)	(20031)
N	(7, 11)	(0111, 1011)	(00111, 01011)	(0011101011)	(03223)
D	(13, 7)	(1101, 0111)	(01101, 00111)	(0110100111)	(12213)
I	(16, 13)	(10000, 1101)	(10000, 01101)	(1000001101)	(20031)

A	(7, 6)	(0111, 0110)	(00111, 00110)	(0011100110)	(03212)

The last column gives the following data matrix of order 5 as follows:

$$B = \begin{bmatrix} 2 & 0 & 0 & 3 & 1 \\ 0 & 3 & 2 & 2 & 3 \\ 1 & 2 & 2 & 1 & 3 \\ 2 & 0 & 0 & 3 & 1 \\ 0 & 3 & 2 & 1 & 2 \end{bmatrix}.$$

Using the spiral traversal form discussed in 2.1.1, we get

$$2 \to 0 \to 0 \to 1 \to 3 \to 0 \to 3 \to 2 \to 2 \to 2 \to 0 \to 0 \to 2 \to 2 \to 1 \to 3 \to 1 \to 0 \to 3 \to 2 \to 3 \to 3 \to 1 \to 1 \to 2.$$

Thus the matrix B is traversed into matrix C as follows:

$$C = \begin{bmatrix} 2 & 2 & 3 & 0 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 2 & 0 & 3 & 3 \\ 2 & 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 & 1 \end{bmatrix}$$

3.2 Encryption: The encryption process is as follows:

(i) Now we choose a lower triangular matrix over $\mathbb{F}_2$ which satisfies the properties discussed in **2** as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

(ii) Let us choose randomly an integer $r = 13$ (say) and obtain

$$A^r = A^{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

This matrix will be made open in the public domain.

(iii) Again, the sender chooses one more integer randomly $s = 7$ (say), which will be kept secret and computes

$$A^{rs} = A^{91} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(iv) Using this matrix as key matrix and the data matrix (obtained from the message M), we obtain the cipher text $(T_1, T_2) = (k, CA^{rs})$, where the column matrix k is the transpose of the first row of A^s.

$$(T_1,\ T_2) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right).$$

These two matrices represent the encrypted message that will travel over the insecure channels and received by the receiver.

3.3 Decryption:

To decrypt the message receiver does the following process.

(i) He separates the first part T_1 of the encrypted message and write $k = (1, 0, 0, 0, 0)$ which is the first row of lower triangular matrix A.

(ii) Now, receiver finds A^{-sr} and further computes C as follows:

$$(C.A^{rs}).A^{-sr} = C = \begin{bmatrix} 2 & 2 & 3 & 0 & 2 \\ 0 & 1 & 3 & 2 & 1 \\ 0 & 2 & 0 & 3 & 3 \\ 2 & 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 & 1 \end{bmatrix}.$$

(iii) Now, using the reverse spiral traversing from 2.1.2 on the matrix C to obtain

$$B=\begin{bmatrix}2&0&0&3&1\\0&3&2&2&3\\1&2&2&1&3\\2&0&0&3&1\\0&3&2&1&2\end{bmatrix}.$$

(iv) Converting each entry of matrix B in binary form as follows:

$$B=\begin{bmatrix}10&00&00&11&01\\00&11&10&10&11\\01&10&10&01&11\\10&00&00&11&01\\00&11&10&01&10\end{bmatrix}.$$

This matrix gives the string of 10 bits as follows:

$$B=\begin{bmatrix}10&00&00&11&01\\00&11&10&10&11\\01&10&10&01&11\\10&00&00&11&01\\00&11&10&01&10\end{bmatrix}\approx\begin{bmatrix}1000001101\\0011101011\\0110100111\\1000001101\\0011100110\end{bmatrix}.$$

(vi) Now convert the string of 10 bits into two tuples each of 5 bits.

$$B=\begin{bmatrix}10&00&00&11&01\\00&11&10&10&11\\01&10&10&01&11\\10&00&00&11&01\\00&11&10&01&10\end{bmatrix}\approx\begin{bmatrix}1000001101\\0011101\ 011\\0110100111\\1000001101\\0011100110\end{bmatrix}\approx\begin{bmatrix}10000&01101\\00111&01011\\01101&00111\\10000&01101\\00111&00110\end{bmatrix}.$$

The corresponding points to these tuples of binary bits are now converted to the tuples in digits form as follows:

(10000, 1101)	(16, 13)
(0111, 1011)	(7, 11)
(1101, 0111)	(13, 7)
(10000, 1101)	(16, 13)
(0111, 0110)	(7, 6)

(vii) The corresponding alphabets to these points are as follows:

$$(16,13)\to I,(7,11)\to N,(13,7)\to D,(7,6)\to A.$$

Arranging these alphabets, we obtain the message **INDIA** which is the original message.

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On Strong Connes Subgroup

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Abstract: Let K be a commutative semiring, R a G –graded K –semialgebra and $R \# K[G]^*$ its smash product. Then using strong Connes subgroup $\tilde{\Gamma}_R$ of G, we prove that $R \# K[G]^*$ is simple iff R is graded simple and R is strongly G-graded and its identity component R_1 is simple.

Key words: Strong Connes Subgroup, graded semirings, smash products and simple semiring.

1. Introduction

This paper is in continuation of [7] in which the validity of results proved by S. Montgomery and D. S. Passman [5] regarding a connection between the Connes subgroup of a group G (which is a purely analogue of the Connes spectrum introduced by A. Connes [3] in the context of action of locally compact groups on Von Neumann algebras) and the ideal structure of a G -graded ring R, its smash product $R \# K[G]^*$ is established. In this paper, R is an additively cancellative semiring. So, R is isomorphic to a subsemiring of the ring of differences R^Δ such that every element of R^Δ is the difference between two elements in the image of R [4]. It is evident from ([4], Proposition 9.42) that there are plenty of such semirings. In R^Δ, we have $a - b = c - d$ if and only if there exist R such that $a + r = c + r'$ and $b + r = d + r'$. The set R^Δ becomes a ring under componentwise addition and multiplication given by $(a - b)(c - d) = (ac + bd) - (ad + bc)$. The zero element of R^Δ is $a - a$, denoted by 0 and multiplicative identity is 1. Clearly, R^Δ contains R by way of embedding $a \mapsto a - 0$ (simply written as a). Another weak version of the condition of having additive inverses, i.e. R being yoked (for R there exists an element r of R such that $a + r = b$ or $b + r = a$) is also required for some results. The ring theoretic results of [8] are studied for such semirings in [10].

We started this paper with the aim to define the strong Connes subgroup for a graded semiring R and relate it with the simplicity of R and its smash product $R \# K[G]^*$. Thus throughout this paper, K be an additively cancellative commutative semiring and R an additively cancellative K–semialgebra graded by a finite group G. If R is a G–graded semiring, then there exists an extension semiring (known as smash product), with same 1, which comes from the study of semi-Hopf algebras. This smash product is denoted by $R \# K[G]^*$, where R is a K–semialgebra. This semiring is a free left R –semimodule with basis $\{p_x \mid x \in G\}$ such that $\sum_{x \in G} p_x = 1$ is decomposition of $1 \in R$ into orthogonal idempotents. Since R and K are additively cancellative, so their rings of differences R^Δ and K^Δ exist. Moreover, if a semiring R is graded by G, then R^Δ becomes a ring graded by G, where

$$(R^\Delta)_g = \{p_g(a-a') \mid a, a' \in R\} = \{a_g - a_g' \mid a_g, a_g' \in R_g\}.$$

Therefore, for $x \in R^\Delta$, $x = a - b \; (a = \sum_{g \in G} a_g, \; b = \sum_{g \in G} b_g \in R)$,

we have the unique representation $a - b = \sum_{g \in G} (a-b)_g$, where $(a-b)_g = a_g - b_g$.

Thus we also have the smash product $R^\Delta \# K^\Delta[G]^*$ which is isomorphic to $(R \# K[G]^*)^\Delta$ (c.f.[5]) and hence $R \# K[G]^*$ embeds in $R^\Delta \# K^\Delta[G]^*$, whereas R embeds in R^Δ. These embeddings are useful to prove the main results of this paper.

2. Some Basic Definitions and Results

The following definitions and results from [4,6-7] are felt to be inseparable part of this paper.

Definition 2.1.

A semiring is a non-empty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(*i*) $(R, +)$ is a commutative monoid with identity element 0;

(*ii*) $(R, .)$ is a monoid with identity element 1;

(*iii*) Multiplication distributes over addition from either side;

(*iv*) $0r = 0 = r0$, for all $r \in R$;

(*v*) $1 \neq 0$.

Definition 2.2.

A nonempty subset (ideal) A of a semiring R is subtractive if and only if $a \in A$ and $a+b \in A$ implies that $b \in A$.

Definition 2.3.

A semiring (semialgebra) R is graded by a finite group G if $R = \sum_{g \in G} \oplus R_g$, where R_g are additive submonoids of R and if $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

Definition 2.4.

For any subset (ideal) I of R, define $I_G = \sum_{g \in G} \oplus (I \cap R_g)$. I is graded if $I = I_G$ Moreover, I_G is the largest graded (ideal) subset of R contained in I.

The following results from [6, 9] of an additively cancellative semiring R will be utilised in subsequent sections:

Lemma 2.5.

Let R be an additively cancellative semiring and R^{Δ} its ring of differences. Let A, B be two nonempty ideals of R and I, J two ideals of R^{Δ} Then:

(*i*) $A^{\Delta}B^{\Delta} = (AB)^{\Delta}$

(*ii*) $A \subseteq B$, then $A^{\Delta} \subseteq B^{\Delta}$. Further, if A is subtractive and $A \subset B$, then $A^{\Delta} \subset B^{\Delta}$.

(*iii*) If $I \subseteq J$, then $I \cap R \subseteq J \cap R$. Further, if R is a yoked semiring and $I \subset J$, then $I \cap R \subset J \cap R$.

(*iv*) $A \subseteq A^{\Delta} \cap R$. Equality holds if A is subtractive;

(*v*) $I \cap R$ is subtractive.

(*vi*) $(I \cap J) \cap R = (I \cap R) \cap (J \cap R)$;

(*vii*) $(I \cap R)\,(J \cap R) \subseteq (IJ) \cap R$;

(*viii*) $(I \cap R)^{\Delta} \subseteq I$. Equality holds if R is a yoked semiring.

Lemma 2.6.

Let R be a semiring graded by G.

(*i*) If A is a graded ideal of R, then A^{Δ} is a graded ideal of R^{Δ}. The converse follows if A is subtractive.

(*ii*) If I is a graded ideal of R^{Δ}, then $I \cap R$ is a graded ideal of R. The converse follows if R is yoked.

Lemma 2.7.

Let R be a semiring graded by a finite group G and A any subset of R. Then for $g \in G$,

(*i*) Each R_g is subtractive;

(*ii*) (*a*) Each A_g is subtractive, if A is a subtractive subset of R;

(*b*) A_G is subtractive, if A is a subtractive submonoid of $(R, +)$;

(*iii*) $(R_g)^{\Delta} = (R^{\Delta})_g$;

(*iv*) $R_g = (R^{\Delta})_g \cap R$;

(*v*) $(A_g)^{\Delta} \subseteq (A^{\Delta})_g$. Equality holds if R is yoked and A is a subtractive submonoid of R;

(*vi*) $(A_G)^{\Delta} \subseteq (A^{\Delta})_G$. Equality holds if R is yoked and A is a subtractive submonoid of R;

(*vii*) Let I be an ideal of R^{Δ} Then (*a*) $(I \cap R)_g = I_g \cap R$;

(*b*) $(I \cap R)_G = I_G \cap R$.

(*viii*) If R is a yoked semiring, then R_1 is a yoked subsemiring of R.

Definition 2.8.

Let P be a graded ideal of R. Then P is graded prime if whenever $A\,B \subseteq P$, where A, B are graded ideals of R, then $A \subseteq P$ or $B \subseteq P$.

We denote the set of all graded left ideals of R by *GrL* and the set of all graded right ideals by *GrR*.

Definition 2.9.

A subsemiring B of R is said to be graded hereditary if $B = AL$ for some $A \in GrR$, $L \in GrL$ and B is a nonnilpotent graded subsemiring of R. The set of all such B is denoted by GrH.

Definition 2.10.

Let R be a G-graded semiring with G finite. We define

$$\Gamma_R = \left\{ x \in G \mid R_{x^{-1}} B_x \text{is nonnilpotent for all } B \in GrH \right\}$$

and

$$\Gamma_{0R} = \left\{ x \in G \mid B_{x^{-1}} B_x \text{is nonnilpotent for all } B \in GrH \right\}.$$

Definition 2.11.

The graded semiring R is said to be strongly graded if $R_g R_h = R_{gh}$ for all $g, h \in G$.

Definition 2.12.

The group G acts on the smash product $R \# K[G]^*$ by $(rp_h)^g = rp_{hg}$ and hence it permutes the ideals of the semiring. We define

$\Lambda = \{x \in G \mid$ for all nonnilpotent ideals of I of $R \# K[G]^*$ we have $I^x I$ nonnilpotent$\}$;

and for each $g \in G$,

$\Lambda_g = \{x \in G \mid$ for all of I of $R \# K[G]^*$, if Ip_gI is nonnilpotent, then I^xI nonnilpotent$\}$.

Definition 2.13.

The semiring R will be called ideal-simple (or simple), if $I = R$, whenever I is an ideal of R such that $I \neq 0$. Similarly, we define graded simple by taking I to be a graded ideal of R.

The correspondence between the ideals of R and $R \# K[G]^*$ is based on the following definitions:

Definition 2.14.

If I is an ideal of R and $x \in G$ we define $I^x = R_{x^{-1}} I R_x \subseteq R$.

Since R_x is an (R, R_1) –bisemimodule we see that I^x is an ideal of R_1. Furthermore $I^1 = I$, $(I^x)^y \subseteq I^{xy}$ and $(I^xJ^x) \subseteq (IJ)^x$.

Remark. Let R be strongly graded semiring and I, J two ideals of R_1 Then $(I^x)^y = I^{xy}$, $I^1 = I$ and $(IJ)^x = I^x J^x$ for $x, y \in G$.

Definition 2.15.

If J is an ideal of R_1, we set

$$\theta(J) = RJp_1R \subseteq R \# K[G]^*$$

and if I is an ideal of $R \subseteq R \# K[G]^*$, we set

$$\varphi(I) = \{a \in R_1 \mid ap_1 \in p_1Ip_1\}.$$

Lemma 2.16.

With the above notation we have

(i) $\theta(J) = (R \# K[G]^*)Jp_1(R \# K[G]^*)$ is an ideal of $R \# K[G]^*$;

(ii) $\varphi(I)$ is an ideal of R_1 and $\varphi(I)p_1 = p_1Ip_1$;

(iii) $\varphi\theta(J) = J$ *so* θ is one-one and φ is onto the set of ideals of R_1;

(iv) If $x \in G$ with $R_x\,\mathrm{R}_{x^{-1}} = \mathrm{R}_{x^{-1}}R_x = R_1$, then $\theta(J^x) = \theta(J)^x$.

Proof. First we prove that $(R \# K[G]^*)p_1 = Rp_1$. Now, let $ap_h \in R \# K[G]^*$, then $ap_hp_1 = ap_1$ (since p_i's are orthogonal idempotents), hence $(R \# K[G]^*)p_1 \subseteq Rp_1$ and $Rp_1 \subseteq (R \# K[G]^*)p_1$ is obvious. Similarly, $p_1(R \# K[G]^*) = p_1R$. Thus, since $Jp_1 = p_1J = p_1\,Jp_1$, because p_1 centralizes R_1, so we have

$$\theta(J) = Rp_1\,Jp_1R = (R \# K[G]^*)Jp_1(R \# K[G]^*)$$

is an ideal of $R \# K[G]^*$, furthermore

$$p_1(R \# K[G]^*)p_1 = p_1Rp_1 = R_1p_1 \cong R_1.$$

Thus $p_1I\,p_1$ is an ideal of R_1p_1. so it follows that $\varphi(I)$ is an ideal of R_1 and

$$p_1Ip_1 = \varphi(I)p_1.$$

This proves parts (i) and (ii) For part third, we have

$$\varphi\theta(J) = p_1\theta(J)p_1 = p_1Rp_1Jp_1Rp_1 = R_1p_1JR_1p_1 = Jp_1.$$

So $\varphi\theta(J) = J$. Finally, let $x \in$ G with

$$R_x\,\mathrm{R}_{x^{-1}} = \mathrm{R}_{x^{-1}}\,R_x = R_1.$$

Then for any $y \in$ G

$$R_{yx^{-1}} = R_{yx^{-1}}R_1 = R_{yx^{-1}}R_x\,\mathrm{R}_{x^{-1}} \subseteq R_x\,\mathrm{R}_{x^{-1}} \subseteq R_{yx^{-1}}$$

So $R_y\,\mathrm{R}_{x^{-1}} = R_{yx^{-1}}$ and hence $\mathrm{RR}_{x^{-1}} = R$. Similarly, we obtain

$$R_xR = Rp_xR = R\,\mathrm{R}_{x^{-1}} = R.$$

Now, using $p_xR_x = R_xp_1$ we have

$$\theta(J^x) = \theta(\mathrm{R}_{x^{-1}}J^xR_x) = R\,\mathrm{R}_{x^{-1}}JR_xp_1R = (R\,\mathrm{R}_{x^{-1}})Jp_x(R_xR) = RJp_1R.$$

But $J \subseteq R_1 \subseteq R$ yields

$$\theta(J)^x = [RJp_1R]^x = RJp_xR = \theta(J^x)$$

Hence (iv) is proved.

Proposition 2.17.

Let R be a strongly G–graded semiring, then the maps θ and φ yield a one to one correspondence between the ideals of R_1 and the ideals of $R \# K[G]^*$. This correspondence preserves inclusion, products and the action of G.

Proof. We know by previous lemma that θ and φ are appropriate maps and that $\varphi\theta(J)=J$ for all ideals J of R_1. Conversely, if I is an ideal of $R \# K[G]^*$, then

$$\theta\varphi(I) = (R \# K[G]^*)\varphi(I)p_1(R \# K[G]^*) = (R \# K[G]^*)p_1Ip_1(R \# K[G]^*)$$

and by ([7], Lemma 4.7),

$$(R \# K[G]^*)p_1Ip_1(R \# K[G]^*) = (R \# K[G]^*)I(R \# K[G]^*) = I$$

Thus θ and φ determines a one to one correspondence between the ideals of R_1 and the ideals of $R \# K[G]^*$, and this correspondence is certainly inclusion preserving. Furthermore if I, I' are two ideals of $R \# K[G]^*$, then

$$\varphi(I)\varphi(I')p_1 = \varphi(I)p_1I'p_1 = p_1Ip_1I'p_1 = p_1I\,I'p_1 = \varphi(II')$$

by ([7], Lemma 4.7). Thus φ preserves products and hence so must θ. Finally by above Lemma, the action of G is preserved.

3. The Strong Connes Subgroup

In order to define the strong Connes subgroup $\widetilde{\Gamma}_R$ of G corresponding to the semiring R first we define:

Definition 3.1.

Let R be a semiring graded by a finite group G. We define

$$Gr\,\widetilde{H} = \{B = AL \mid A \in GrR,\ L \in GrL\}.$$

Thus the nonnilpotent members of $Gr\widetilde{H}$ are precisely the graded hereditary of R. Clearly, $GrH \subseteq Gr\widetilde{H}$. The following definitions and results are same as in ring theory and can be proved in the same way.

$$\widetilde{\Gamma}_R = \{g \in G \mid L_{xg}A_{g^{-1}y} = L_xA_y \text{ for all } A \in GrR, L \in GrL \text{ and } x, y \in G\}$$

and

$$\widetilde{\Gamma}_{0R} = \{g \in G \mid B_{xg}C_{g^{-1}y} = B_xC_y \text{ for all } B, C \in Gr\widetilde{H} \text{ and } x, y \in G\}$$

Also, we define

$$Gr\widetilde{H}_\Delta = \{B = AL \mid A \in GrR_\Delta, L \in GrL_\Delta\},$$

$$\widetilde{\Gamma}_\Delta = \{g \in G \mid L_{xg}A_{g^{-1}y} = L_xA_y \text{ for all } A \in GrR_\Delta, L \in GrL_\Delta \text{ and } x, y \in G\}$$

and

$$\widetilde{\Gamma}_{0\Delta} = \{g \in G \mid B_{xg}C_{g^{-1}y} = B_xC_y \text{ for all } B, C \in Gr\widetilde{H}_\Delta \text{ and } x, y \in G\}$$

Lemma 3.2.

$\widetilde{\Gamma}_R$ is a subgroup of *G* and $\widetilde{\Gamma}_{0R} = \widetilde{\Gamma}_R \subseteq \Gamma_R$.

Definition 3.3.

We define

$$\widetilde{\Lambda} = \left\{ x \in G \;\middle|\; I^x = I \text{ for all I ideal of R \# K [G]* and for each } g \in \mathrm{G} \right\}$$

and

$$\widetilde{\Lambda}_g = \left\{ x \in G \;\middle|\; (Ip_g I')^x = Ip_g I' \text{ for all left ideal I of R \# K [G]* and I' ideal of R \# K [G]*} \right\}$$

Lemma 3.4.

$\widetilde{\Lambda}$ is a normal subgroup, each $\widetilde{\Lambda}_g$ is a subgroup of *G* and $\widetilde{\Lambda} = \cap_{g \in G} \widetilde{\Lambda}_g$.

Theorem 3.5 .

Let *R* be a semiring graded by a finite group *G*. Then for all *g* ∈ *G* we have $\widetilde{\Lambda}_g = \widetilde{\Gamma}^g$ and hence $\widetilde{\Lambda} = \cap_{g \in G} \widetilde{\Gamma}_g$.

Proof. Let *g* ∈ *G* and $A \in GrR$, $L \in GrL$. Then by ([7], Lemma 4.13), $I = Lp_g$ is a left ideal of *R # K* [*G*]* and $I' = p_g A$ is a right ideal of *R # K* [*G*]* and we have

$$Ip_g I' = (Lp_g)p_g(p_g A) = Lp_g A.$$

Conversely, let *I* be a left ideal of *R # K* [*G*]* and *I*$'$ a right ideal of *R # K* [*G*]*. Then by ([7], Lemma 4.13), there exist $A \in GrR$ and $L \in GrL$ with

$$Ip_g = Lp_g \quad \text{and} \quad p_g I' = p_g A.$$

Thus

$$Ip_g I' = Ip_g x p_g I' = Lp_g A$$

or we can say that the ideals of *R # K* [*G*] * are of the form $Ip_g I' = Lp_g A$, where $A \in GrR$, $L \in GrL$. Now, if *l*, *h*, $x \in G$, then

$$p_{lg}\left(Lp_g A\right)p_{m^{-1}g} = L_{lg\,g^{-1}}A_{gg^{-1}m}p_{m^{-1}g} = L_l A_m p_{m^{-1}g}$$

and

$$p_{lg}\left(Lp_{gx}A\right)p_{m^{-1}g} = L_{l\left(gx^{-1}g^{-1}\right)}A_{\left(gxg^{-1}\right)m}p_{m^{-1}g}.$$

Thus by letting *l*, *m* vary we see that

$$p_{lg}(Lp_g A)p_{m^{-1}g} = p_{lg}(Lp_{gx}A)p_{m^{-1}g}$$

iff

$$L_l A_m p_{m^{-1}g} = L_{l(gx^{-1}g^{-1})}A_{(gxg^{-1})m}p_{m^{-1}g}$$

implies when we vary *l* over *G,* we get

$$\left(Lp_g A\right)p_{m^{-1}g} = \left(Lp_{gx}A\right)p_{m^{-1}g}$$

iff

$$L_l A_m p_{m^{-1}g} = L_{l\left(gx^{-1}g^{-1}\right)}A_{\left(gxg^{-1}\right)m}p_{m^{-1}g}.$$

This implies $Lp_g A = Lp_{gx}A$, iff $L_l A_m = L_{l\left(gx^{-1}g^{-1}\right)}A_{\left(gxg^{-1}\right)m}$ for all $l, m \in G.$

Since $(Lp_g A)^x = Lp_{gx}A,$ it follows from the above that $x \in \widetilde{\Lambda}_g$ iff $gxg^{-1} \in \widetilde{\Gamma}$ and hence iff $x \in \widetilde{\Gamma}_g$. Thus $\widetilde{\Lambda}_g = \widetilde{\Gamma}_g$ and Lemma 3.4, yields the result.

Note 3.6.

We know by Lemma 2.16, that if *J* is an ideal of R_1, then $\theta(J)^x = RJp_1R$ is an ideal of $R\#K[G]^*$ Furthermore θ is one - one on the set of these ideals and if $g \in G$ with $R_x\ \mathrm{R}_{x^{-1}} = \mathrm{R}_{x^{-1}}\,R_x = R_1$, then $\theta(J^x) = \theta(J)^x$.

Proposition 3.7.

Let $\widetilde{\Gamma}_R$ be the strong Connes subgroup of *G*.

(*i*) $R_{\widetilde{\Gamma}_R} = \sum_{g\in\widetilde{\Gamma}_R} R_g$ is strongly $\widetilde{\Gamma}_R$ –graded;

(*ii*) $J^g = J$ for all $g \in \widetilde{\Gamma}_R$ and all *J* ideal of R_1;

(iii) $I^g = I$ for all $g \in \widetilde{\Gamma}_R$ and all I ideal of $R \# K[G]^*$ of the form $I = \theta(J)$ for some J an ideal of R_1.

Proof. Let $g \in \widetilde{\Gamma}_R$ Since $R \in GrH$, we have

$$R_{g^{-1}} R_g = R_{g^{-1}g} R_{g^{-1}g} = R_1^2 = R_{1.}$$

It then follows that $R_g R_h = R_{gh}$ for all $g, h \in \widetilde{\Gamma}_R$, so ($i$) is proved.

Let $g \in \widetilde{\Gamma}_R$, $A \in GrR$ and $L \in GrL.$ Since L, A and R are all contained in $Gr\,\widetilde{H}$, so by Lemma 3.2, we have

$$R_{g^{-1}} L_1 = R_{g^{-1}g} L_{g^{-1}} = R_1 L_{g^{-1}} = L_{g^{-1}} \text{ and } \quad A_1 R_g = A_g R_{g^{-1}g} = A_g R_1 = R_{1.}$$

Furthermore,

$$L_{g^{-1}} A_g = L_{g^{-1}g} A_{g^{-1}g} = L_1 A_1.$$

Thus,

$$(L_1 A_1)^g = R_{g^{-1}} L_1 A_1 R_g = L_{g^{-1}} A_g = L_1 A_1.$$

Since any ideal J of R_1 is of the form L_1A_1, it follows that $J^g = J$ for all $g \in \widetilde{\Gamma}_R$. Thus ($ii$) and Lemma 2.16 ($iv$) yields ($iii$).

Corollary 3.8.

Let R be a semiring graded by a finite group G with $\widetilde{\Gamma}_R$ the strong Connes subgroup of G.

(i) If R is strongly graded, then

$$\widetilde{\Lambda} = \widetilde{\Gamma}_R = \{g \in G | J^g = J \text{ for all } J \text{ ideal of } R_1\}$$

(ii) $\widetilde{\Gamma}_R = \mathrm{G}$ if and only if R is strongly G–graded and all ideals of R_1 are G–stable.

Proof. If R is strongly graded, then by Proposition 2.17, all ideals of $R \# K[G]^*$ are of the form $\theta(J)$ for some J ideal of R_1. Then by Proposition 3.7(iii), we see that $\widetilde{\Gamma}_R \subseteq \Lambda.$ Now by Theorem 3.5, $\widetilde{\Lambda}_g = \widetilde{\Gamma}_g$ Finally by [7], Lemma 4.8, we conclude that $\widetilde{\Lambda}$ is the stabilizer of all ideals of R_1

since it is defined to be the stabilizer of all ideals of $R \# K[G]^*$. Thus (*i*) is proved. Part (*ii*), follows immediately from this and Proposition 3.7 (*ii*).

Lemma 3.9.

Let R *be a semiring graded by a finite group* G *and* R^Δ *its ring of differences.*

(*i*) If R^Δ is graded simple, then R is graded simple;

(*ii*) If R is graded simple and yoked, then R^Δ is graded simple.

Proof. **(*i*)** Suppose R is not graded simple and $0 \neq I$ is a graded ideal of R, then there exist $0 \neq I^\Delta$ a graded ideal of R^Δ by Lemma 2.6, which is a contradiction to the fact that R^Δ is graded simple. Hence R is graded simple.

(*ii*) Suppose R^Δ is not graded simple and there exist a nonzero graded ideal $I \neq 0$ in R^Δ, then by Lemma 2.6, $I \cap R$ is a graded ideal of R. Also by, Lemma 2.5(*viii*), if R is yoked, then

$$(I \cap R)^\Delta = I \neq 0$$

implies $I \cap R \neq 0$. Thus our supposition is wrong and hence R^Δ is graded simple.

Lemma 3.10.

Let R be a graded, yoked and additively cancellative semiring and R^Δ its ring of differences.

(*i*) If A is an additive subgroup of R^Δ, then $(I \cap R)^\Delta = I$

(*ii*) $\widetilde{\Gamma}_R = \widetilde{\Gamma}_\Delta$.

Proof. (*i*) The inclusion $(I \cap R)^\Delta \subseteq I$ is obvious. Conversely, let $a - b \in I$. Let R be a yoked semiring and note that for an additively cancellative yoked semiring, either $a–b \in R$ or $b–a \in R$, for all $a - b \in R^\Delta$. Thus, either $a - b \in I \cap R$ or $b - a \in I \cap R$. In any case $a - b \in (I \cap R)^\Delta$ implying $I \subseteq (I \cap R)^\Delta$.

(*ii*) Let $g \in \widetilde{\Gamma}_R$ and J be an ideal of R^Δ. Then by Lemma 2.6, $J \cap R$ is an ideal of R.

Since $A \cap R \in Gr\, R$, $L \cap R \in GrL$, by definition of $\widetilde{\Gamma}_R$, we get

$$(L \cap R)_{xg}(A \cap R)_{g^{-1}y} = (L \cap R)_x (A \cap R)_y \quad \text{for all } x, y \in G$$

By Lemma 2.7(*viii*)(*a*) we get

$$(L_{xg} \cap R)(A_{g^{-1}y} \cap R) = (L_x \cap R)(A_y \cap R).$$

Now by using Lemma 2.7(*viii*)(*a*) & (*i*), we have

$$\begin{aligned} L_{xg} A_{g^{-1}y} &= (L_{xg} \cap R)^\Delta (A_{g^{-1}y} \cap R)^\Delta \\ &= (L_x \cap R)^\Delta (A_y \cap R)^\Delta = L_x A_y. \end{aligned}$$

Hence $\widetilde{\Gamma}_R \subseteq \widetilde{\Gamma}_\Delta$.

Theorem 3.11.

Let R be graded semiring. Then the following are equivalent:

(i) $R \# K[G]^*$ is simple;

(ii) R is graded simple and $\widetilde{\Gamma}_R = G$;

(iii) R is graded simple and $\Gamma_R = G$;

(iv) R is strongly $G-$graded and R_1 is simple.

Proof. (i) $\Rightarrow$ (ii) We know if $R \# K[G]^*$ is simple, then $R \# K[G]^*$ is prime, so obviously $\widetilde{\Lambda} = G$ and hence $\widetilde{\Gamma}_R = G$. Furthermore, if $J \neq 0$ is a graded ideal of R, then by ([7], Lemma 4.13(ii)), $J.(R \# K[G]^*)$ is an ideal of $R \# K[G]^*$. Thus $J.(R \# K[G]^*) = R \# K[G]^*$, so

$$Rp_1 = (R \# K[G]^*)p_1 = J.(R \# K[G]^*)p_1 = Jp_1 \text{ and } J = R.$$

(ii) $\Rightarrow$ (i) Since R is graded simple implies R^Δ is graded simple. Also by above Lemma, we have $\widetilde{\Gamma}_R \supseteq \widetilde{\Gamma}_R = G$ implies $\widetilde{\Gamma}_\Delta = G.$ Now, R^Δ is graded simple implies $R^\Delta \# K^\Delta[G]^*$ is simple and hence

$$R^\Delta \neq K[G]^\Delta \cong (R \# K[G]^*)^\Delta$$

is simple. So, by Lemma 3.9, $R \# K[G]^*$ is simple.

(ii) $\Rightarrow$ (iii) By Lemma 3.9, if R is graded simple and yoked, then R^Δ is graded simple. Also $\widetilde{\Gamma}_\Delta \supseteq \widetilde{\Gamma}_R = G$ implies $\widetilde{\Gamma}_\Delta = G,$ so by Corollary 3.6,[5], $\Gamma_\Delta = G$. Now, R^Δ is graded simple implies R^Δ is graded prime and $\Gamma_\Delta = G$. This implies

$$R^\Delta \neq K[G]^\Delta \cong (R \# K[G]^*)^\Delta$$

is prime. So by Lemma 3.9, $R \# K[G]^*$ is prime implies R is graded simple and $\Gamma_R = G$.

(iii) $\Rightarrow$ (iv) By Lemma 3.9, if R is graded simple and yoked, then R^Δ is graded simple. Also by above Lemma, we have $\widetilde{\Gamma}_\Delta \supseteq \widetilde{\Gamma}_R = G$ implies $\Gamma_\Delta = G$, so by corollary 3.6[5] $\Gamma_\Delta = G$. Thus by corollary 3.6. [5] R^Δ is strongly graded and $(R^\Delta)_1$ is simple implies$(R^\Delta)_1 \cong (R_1)^\Delta$ is simple. Hence by Lemma R_1 is simple.

(iv) $\Rightarrow$ (ii) follows in the similar manner as (ii) $\Rightarrow$ (iii).

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CONSTRUCTION OF MDS RHOTRICES USING SPECIAL TYPE OF CIRCULANT RHOTRICES OVER FINITE FIELDS

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ABSTRACT

Maximum distance separable (MDS) matrices play an important role in the designing of block ciphers and hash functions in cryptography. Circulant matrices have wide applications in control theory, graph theory and in solution of linear equations. We introduce a special type of circulant rhotrices. Using these rhotrices, we construct MDS rhotrices over finite fields.

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1. INTRODUCTION

Maximum Distance separable matrices have wide range of applications in different areas of mathematical sciences, computer sciences and other sciences. MDS matrix has diffusion properties that are used in block ciphers and cryptographic hash functions. Several researchers have designed block ciphers such as AES[4, 6], AES-MDS[17] and some hash functions Maelstrom[7], Grostl [8] using MDS matrices. MDS matrices provide security against different cryptanalysis [5, 14-16]. There are several methods to construct MDS matrices. Sajadieh et al.[9] and Lacan and Fimes[10] used Vandermonde matrices for the construction of MDS matrices while Youssef et al.[35] used Cauchy matrices.Guo et al. [9], Gupta and Ray [10] used companion matrices for the construction of MDS matrices. Sajadieh et al.[18, 19] and Wu et al. [34] proposed new $d \times d$ MDS matrices based on companion matrices for smaller values of d. Circulant matrices are also used for the construction of MDS matrices. Junod et al. [12] constructed new class of MDS matrices whose submatrices were circulant matrices. Gupta and Ray [11] constructed MDS matrices from circulant-like matrices.

Rhotrix is a mathematical object which is in some way between 2×2-

dimensional and 3×3 - dimensional matrices introduced in 2003 by Ajibade [2]. A rhotrix of dimension 3 is defined as

$$R_3 = \left\langle \begin{array}{ccc} & a_1 & \\ a_2 & a_3 & a_4 \\ & a_5 & \end{array} \right\rangle, \qquad (1.1)$$

where a_1, a_2, a_3, a_4, a_5 are real numbers. Algebra and analysis of rhotrices is discussed in the literature, see [1, 3, 13, 20-33]. Sharma and Kumar [25] introduced companion rhotrix to construct MDS rhotrices. In the present paper, we introduce circulant rhotrix and special type of circulant rhotrix. Further, weconstruct MDS rhotrices using special type of circulant rhotrix over GF (2^8) .

2. MAIN RESULTS

Definition 2.1: A circulant rhotrix R_n is defined as

$$R_n = \left\langle \begin{array}{ccccccccccc}
 & & & & & a_0 & & & & & \\
 & & & & a_d & b_0 & a_1 & & & & \\
 & & & a_{d-1} & b_{d-1} & a_0 & b_1 & . & & & \\
 & & . & . & a_d & b_0 & . & . & . & & \\
 & . & . & . & . & . & . & . & . & . & \\
a_1 & b_1 & . & . & . & . & . & . & . & b_{d-1} & a_d \\
 & a_2 & b_2 & . & . & . & . & . & b_{d-2} & a_{d-1} & \\
 & & . & . & . & . & . & . & a_{d-2} & & \\
 & & & . & . & . & . & . & & & \\
 & & & & . & b_0 & . & & & & \\
 & & & & & a_0 & & & & &
\end{array} \right\rangle \qquad (2.1)$$

where $a_i, b_j; i = 1,..,d; j = 1,..,d-1$ are real numbers. It is also denoted as $Cir((a_0,............,a_d),(b_0,............,b_{d-1}))$.

For example: Circulant rhotrix of order seven can be presented as

$$
R_7 = \left\langle \begin{array}{ccccccc}
 & & & a_0 & & & \\
 & & a_3 & b_0 & a_1 & & \\
 & a_2 & b_2 & a_0 & b_1 & a_2 & \\
a_1 & b_1 & a_3 & b_0 & a_1 & b_2 & a_3 \\
 & a_2 & b_2 & a_0 & b_1 & a_2 & \\
 & & a_3 & b_0 & a_1 & & \\
 & & & a_0 & & &
\end{array} \right\rangle . \tag{2.2}
$$

Definition 2.2: A special type of circulant rhotrix C_n is defined as

$$
C_n = \left\langle \begin{array}{ccccccccccc}
 & & & & & a & & & & & \\
 & & & & 1 & b_0 & 1 & & & & \\
 & & & 1 & b_{d-1} & a_0 & b_1 & 1 & & & \\
 & & . & . & a_{d-1} & b_0 & . & . & . & & \\
 & . & . & . & . & a_0 & . & . & . & . & \\
1 & b_1 & . & . & . & . & . & . & . & b_{d-1} & 1 \\
 & a_1 & b_2 & . & . & . & . & . & b_{d-2} & a_{d-1} & \\
 & & . & . & . & . & . & . & a_{d-2} & & \\
 & & & . & . & . & . & . & & & \\
 & & & & . & b_0 & . & & & & \\
 & & & & & a_0 & & & & &
\end{array} \right\rangle \tag{2.3}
$$

where $a, a_i, b_j; i, j = 1, 2, .., d-1$ are real numbers. It is also denoted as $[(a, cir(a_0, ..., a_{d-1})), cir(b_0, ..., b_{d-1})]$.

Theorem 2.3 Let R_7 be a special type of circulant rhotrix and $R_1 = (a^{-1}, cir(1, a^{-1}, a))$ and $R_2 = cir(1, 1+a, a^{-1})$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_7^3 of order 7.

Proof: Let

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & R_1^3[1][1] & & & \\ & & R_1^3[2][1] & R_2^3[1][1] & R_1^3[1][2] & & \\ & R_1^3[3][1] & R_2^3[2][1] & R_1^3[2][2] & R_2^3[1][2] & R_1^3[1][3] & \\ R_1^3[4][1] & R_2^3[3][1] & R_1^3[3][2] & R_2^3[2][2] & R_1^3[2][3] & R_2^3[1][3] & R_1^3[1][4] \\ & R_1^3[4][2] & R_2^3[3][2] & R_1^3[3][3] & R_2^3[2][3] & R_1^3[2][4] & \\ & & R_1^3[4][3] & R_2^3[3][3] & R_1^3[3][4] & & \\ & & & R_1^3[4][4] & & & \end{array} \right\rangle \quad (2.4)$$

We consider $R_1 = (a^{-1}, cir(1, a^{-1}, a))$ in (2.4), therefore we have

$$R_1^3 = \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-1} & a \\ 1 & a & 1 & a^{-1} \\ 1 & a^{-1} & a & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-1} & a \\ 1 & a & 1 & a^{-1} \\ 1 & a^{-1} & a & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-1} & a \\ 1 & a & 1 & a^{-1} \\ 1 & a^{-1} & a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a + a^{-3} + a^{-1} + 1 & a^2 + a^{-2} + a^{-1} + 1 & a^2 + a^{-2} + a^{-1} + 1 & a^2 + a^{-2} + a^{-1} + 1 \\ a^2 + a^{-2} + a^{-1} + 1 & a^3 + a^{-3} + a^{-1} + 1 & a^2 + a^{-1} & a^{-2} + a^{-1} \\ a^2 + a^{-2} + a^{-1} + 1 & a^{-2} + a^{-1} & a^3 + a^{-3} + a^{-1} + 1 & a^2 + a^{-1} \\ a^2 + a^{-2} + a^{-1} + 1 & a^2 + a^{-1} & a^{-2} + a^{-1} & a^3 + a^{-3} + a^{-1} + 1 \end{pmatrix} \quad (2.5)$$

Here, a is the root of $a^8 + a^4 + a^3 + a^2 + 1$. Therefore, $a^{-1} = a^7 + a^3 + a^2 + a$, $a^{-2} = a^6 + a^2 + a + 1$ and $a^{-3} = a^7 + a^5 + a^3 + a^2 + 1$. This gives,

$R_1^3[1][1] = a + a^{-3} + a^{-1} + 1 = a^5 \neq 0$,

$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = a^2 + a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 + a^2 \neq 0$

$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = a^2 + a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 + a^2 \neq 0$

$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = a^3 + a^{-3} + a^{-1} + 1 = a^5 + a^3 + a + 1 \neq 0$

$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][2] = a^2 + a^{-1} = a^7 + a^3 + a \neq 0$

$R_1^3[2][4] = R_1^3[3][2] = R_1^3[4][3] = a^{-2} + a^{-1} = a^7 + a^6 + a^3 + 1 \neq 0.$

Hence, R_1^3 is MDS. Now,

$$R_2^3 = \begin{pmatrix} 1 & 1+a & a^{-1} \\ a^{-1} & 1 & 1+a \\ 1+a & a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1+a & a^{-1} \\ a^{-1} & 1 & 1+a \\ 1+a & a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1+a & a^{-1} \\ a^{-1} & 1 & 1+a \\ 1+a & a^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-3}+a^3+a^2+a & a^{-2}+a^{-1}+1 & a^2+a^{-1}+1 \\ a^2+a^{-1}+1 & a^{-3}+a^3+a^2+a & a^{-2}+a^{-1}+1 \\ a^{-2}+a^{-1}+1 & a^2+a^{-1}+1 & a^{-3}+a^3+a^2+a \end{pmatrix}. \tag{2.6}$$

The matrix (2.6) gives,

$$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = a^3 + a^{-3} + a^2 + 1 = a^7 + a^5 + a + 1 \neq 0$$

$$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][1] = a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 \neq 0$$

$$R_2^3[1][3] = R_2^3[3][2] = R_2^3[2][1] = a^2 + a^{-2} + 1 = a^6 + a \neq 0.$$

Therefore, R_2^3 is MDS.

From (2.4)-(2.6), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & a^5 & & & \\ & & a^7+a^6+a^3+a^2 & a^7+a^5+a+1 & a^7+a^6+a^3+a^2 & & \\ & a^7+a^6+a^3+a^2 & a^6+a & a^5+a^3+a+1 & a^7+a^6+a^3 & a^7+a^6+a^3+a^2 & \\ a^7+a^6+a^3+a^2 & a^7+a^6+a^3 & a^7+a^6+a^3+1 & a^7+a^5+a+1 & a^7+a^3+a & a^6+a & a^7+a^6+a^3+a^2 \\ & a^7+a^3+a & a^6+a & a^5+a^3+a+1 & a^7+a^6+a^3 & a^7+a^6+a^3+1 & \\ & & a^7+a^6+a^3+1 & a^7+a^5+a+1 & a^7+a^3+a & & \\ & & & a^5+a^3+a+1 & & & \end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

∎

Theorem2.4 Let R_7 be a special type of circulant rhotrix and $R_1 = (a, cir(1, 1+a^{-1}, 1+a))$ and $R_2 = cir(1, a^{-1}, a+a^{-1})$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8+a^4+a^3+a^2+1$. Then R_1^3 and R_2^3 form MDS rhotrix R_7^3 of order 7.

Proof: Let R_7^3 be defined as in (2.4) and $R_1 = (a, cir(1, 1+a^{-1}, 1+a))$ in (2.4), therefore we have

$$
R_1^3=\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a^{-1} & 1+a \\ 1 & 1+a & 1 & 1+a^{-1} \\ 1 & 1+a^{-1} & 1+a & 1 \end{pmatrix}\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a^{-1} & 1+a \\ 1 & 1+a & 1 & 1+a^{-1} \\ 1 & 1+a^{-1} & 1+a & 1 \end{pmatrix}\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a^{-1} & 1+a \\ 1 & 1+a & 1 & 1+a^{-1} \\ 1 & 1+a^{-1} & 1+a & 1 \end{pmatrix}
$$

$$
=\begin{pmatrix} a+a^3+a^{-1}+1 & a^2+a^{-2}+a+1 & a^2+a^{-2}+a+1 & a^2+a^{-2}+a+1 \\ a^2+a^{-2}+a+1 & a^3+a^2+a^{-3}+a^{-2}+a^{-1}+1 & a^2+a^{-2}+1 & a^2+a+a^{-2}+a^{-1}+1 \\ a^2+a^{-2}+a+1 & a^2+a+a^{-2}+a^{-1}+1 & a^3+a^2+a^{-3}+a^{-2}+a^{-1}+1 & a^2+a^{-2}+1 \\ a^2+a^{-2}+a+1 & a^2+a^{-2}+1 & a^2+a+a^{-2}+a^{-1}+1 & a^3+a^2+a^{-3}+a^{-2}+a^{-1}+1 \end{pmatrix} \qquad (2.7)
$$

This gives,

$$R_1^3[1][1]=a+a^3+a^{-1}+1=a^7+a^2+1\neq 0,$$

$$R_1^3[1][2]=R_1^3[1][3]=R_1^3[1][4]=a^2+a^{-2}+a+1=a^6\neq 0$$

$$R_1^3[2][1]=R_1^3[3][1]=R_1^3[4][1]=a^2+a^{-2}+a+1=a^6\neq 0$$

$$R_1^3[2][2]=R_1^3[3][3]=R_1^3[4][4]=a^3+a^2+a^{-3}+a^{-2}+a^{-1}+1$$
$$=a^6+a^5+a^3+1\neq 0$$

$$R_1^3[2][3]=R_1^3[3][4]=R_1^3[4][2]=a^2+a^{-2}+1=a^6+a\neq 0$$

$$R_1^3[2][4]=R_1^3[3][2]=R_1^3[4][3]=a^{-2}+a^{-1}+a^2+a+1=a^7+a^6+a^3+a^2+a\neq 0.$$

Hence, R_1^3 is MDS. Now,

$$
R_2^3=\begin{pmatrix} 1 & a^{-1} & a+a^{-1} \\ a+a^{-1} & 1 & a^{-1} \\ a^{-1} & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1} & a+a^{-1} \\ a+a^{-1} & 1 & a^{-1} \\ a^{-1} & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-1} & a+a^{-1} \\ a+a^{-1} & 1 & a^{-1} \\ a^{-1} & a+a^{-1} & 1 \end{pmatrix}
$$

$$
=\begin{pmatrix} a^3+a^{-1}+a+1 & a^{-3}+a^{-2}+a^2 & a^{-3}+a^{-2}+a^{-1} \\ a^{-3}+a^{-2}+a^{-1} & a^3+a^{-1}+a+1 & a^{-3}+a^{-2}+a^2 \\ a^{-3}+a^{-2}+a^2 & a^{-3}+a^{-2}+a^{-1} & a^3+a^{-1}+a+1 \end{pmatrix}. \qquad (2.8)
$$

The matrix (2.8) gives,

$$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = a^3 + a^{-1} + a + 1 = a^7 + a^2 + 1 \neq 0$$

$$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][1] = a^{-3} + a^{-2} + a^2 = a^7 + a^6 + a^5 + a^3 + a^2 + a \neq 0$$

$$R_2^3[1][3] = R_2^3[3][2] = R_2^3[2][1] = a^{-3} + a^{-2} + a^{-1} = a^6 + a^5 + a^2 \neq 0.$$

Therefore, R_2^3 is MDS.

From (2.7) and (2.8), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & a^7+a^2+1 & & & \\ & & a^6 & a^7+a^2+1 & a^6 & & \\ & a^6 & a^7+a^6+a^5+a^3+a^2+a & a^6+a^5+a^3+1 & a^7+a^6+a^5+a^3+a^2+a & a^6 & \\ a^6 & a^6+a^5+a^2 & a^7+a^6+a^3+a^2+a & a^7+a^2+1 & a^6+a & a^6+a^5+a^2 & a^6 \\ & a^6+a & a^7+a^6+a^5+a^3+a^2+a & a^6+a^5+a^3+1 & a^7+a^6+a^3 & a^7+a^6+a^3+a^2+a & \\ & & a^7+a^6+a^3+a^2+a & a^7+a^2+1 & a^6+a & & \\ & & & a^6+a^5+a^3+1 & & & \end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

■

Theorem 2.5 Let R_7 be a special type of circulant rhotrix and $R_1 = (a+1, cir(1, a+a^{-1}, 1+a))$ and $R_2 = cir(1, a, 1+a^{-1})$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_7^3 of order 7.

Proof: Let R_7^3 be defined as in (2.4) and $R_1 = (a+1, cir(1, a+a^{-1}, 1+a))$ in (2.4), therefore we have

$$R_1^3 = \begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a+a^{-1} & 1+a \\ 1 & 1+a & 1 & a+a^{-1} \\ 1 & a+a^{-1} & 1+a & 1 \end{pmatrix} \begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a+a^{-1} & 1+a \\ 1 & 1+a & 1 & a+a^{-1} \\ 1 & a+a^{-1} & 1+a & 1 \end{pmatrix} \begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a+a^{-1} & 1+a \\ 1 & 1+a & 1 & a+a^{-1} \\ 1 & a+a^{-1} & 1+a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a+a^2+a^3+a^{-1}+1 & a^2+a^{-2}+a^{-1}+1 & a^2+a^{-2}+a^{-1}+1 & a^2+a^{-2}+a^{-1}+1 \\ a^2+a^{-2}+a^{-1}+1 & a^2+a+a^{-3}+a^{-1}+1 & a^3+a^{-2} & a^3+a^2+a^{-2}+a^{-1} \\ a^2+a^{-2}+a^{-1}+1 & a^3+a^2+a^{-2}+a^{-1} & a^2+a+a^{-3}+a^{-1}+1 & a^3+a^{-2} \\ a^2+a^{-2}+a^{-1}+1 & a^3+a^{-2} & a^3+a^2+a^{-2}+a^{-1} & a^2+a+a^{-3}+a^{-1}+1 \end{pmatrix} \quad (2.9)$$

This gives,

$$R_1^3[1][1] = a + a^2 + a^3 + a^{-1} + 1 = a^7 + 1 \neq 0\,,$$

$$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = a^2 + a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 + a^2 \neq 0,$$

$$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = a^2 + a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 + a^2 \neq 0,$$

$$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = a^2 + a^{-3} + a^{-1} + a + 1 = a^5 + a^2 \neq 0,$$

$$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][2] = a^3 + a^{-2} = a^6 + a^3 + a^2 + a + 1 \neq 0,$$

$$R_1^3[2][4] = R_1^3[3][2] = R_1^3[4][3] = a^{-2} + a^{-1} + a^3 + a^2 = a^7 + a^6 + a^2 + 1 \neq 0.$$

Hence, R_1^3 is MDS. Now,

$$R_2^3 = \begin{pmatrix} 1 & a & 1+a^{-1} \\ 1+a^{-1} & 1 & a \\ a & 1+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 1+a^{-1} \\ 1+a^{-1} & 1 & a \\ a & 1+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 1+a^{-1} \\ 1+a^{-1} & 1 & a \\ a & 1+a^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^3 + a^{-3} + a^{-2} + a^{-1} & a^2 + a^{-2} + 1 & a^2 + a + 1 \\ a^2 + a + 1 & a^3 + a^{-3} + a^{-2} + a^{-1} & a^2 + a^{-2} + 1 \\ a^2 + a^{-2} + 1 & a^2 + a + 1 & a^3 + a^{-3} + a^{-2} + a^{-1} \end{pmatrix}. \tag{2.10}$$

The matrix (2.10) gives,

$$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = a^3 + a^{-3} + a^2 + a = a^7 + a^5 + a + 1 \neq 0$$

$$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][1] = a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 \neq 0$$

$$R_2^3[1][3] = R_2^3[3][2] = R_2^3[2][1] = a^{-2} + a^{-2} + 1 = a^6 + a \neq 0.$$

Therefore, R_2^3 is MDS. From (2.9) and (2.10), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & a^7+1 & & & \\ & & a^7+a^6+a^3+a^2 & a^6+a^5+a^3+a^2 & a^7+a^6+a^3+a^2 & & \\ & a^7+a^6+a^3+a^2 & a^2+a+1 & a^5+a^2 & a^6+a & a^7+a^6+a^3+a^2 & \\ a^7+a^6+a^3+a^2 & a^6+a & a^7+a^6+a^2+1 & a^6+a^5+a^3+a^2 & a^6+a^3+a^2+a+1 & a^2+a+1 & a^7+a^6+a^3+a^2 \\ & a^6+a^3+a^2+a+1 & a^2+a+1 & a^5+a^2 & a^6+a & a^7+a^6+a^2+1 & \\ & & a^7+a^6+a^2+1 & a^6+a^5+a^3+a^2 & a^6+a^3+a^2+a+1 & & \\ & & & a^5+a^2 & & & \end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

∎

Theorem 2.6 Let R_7 be a special type of circulant rhotrix and $R_1 = (a+1, cir(1, a, a^2+a))$ and $R_2 = cir(1, a, 1+a^2)$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_7^3 of order 7.

Proof: Let R_7^3 be defined as in (2.4) and $R_1 = (a+1, cir(1, a, a^2+a))$ in (2.4), therefore we have

$$R_1^3 = \begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a & a^2+a \\ 1 & a^2+a & 1 & a \\ 1 & a & a^2+a & 1 \end{pmatrix}\begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a & a^2+a \\ 1 & a^2+a & 1 & a \\ 1 & a & a^2+a & 1 \end{pmatrix}\begin{pmatrix} a+1 & 1 & 1 & 1 \\ 1 & 1 & a & a^2+a \\ 1 & a^2+a & 1 & a \\ 1 & a & a^2+a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^3+a & a^4+a^3+a & a^4+a^3+a & a^4+a^3+a \\ a^4+a^3+a & a^6+a^5+a^4+a & a^3+a^2+1 & a^5+a^3+1 \\ a^4+a^3+a & a^5+a^3+1 & a^6+a^5+a^4+a & a^3+a^2+1 \\ a^4+a^3+a & a^3+a^2+1 & a^5+a^3+1 & a^6+a^5+a^4+a \end{pmatrix}, \qquad (2.11)$$

which clearly shows that all the entries in the matrix are non-zero. Hence, R_1^3 is MDS. Now,

$$R_2^3 = \begin{pmatrix} 1 & a & 1+a^2 \\ 1+a^2 & 1 & a \\ a & 1+a^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 1+a^2 \\ 1+a^2 & 1 & a \\ a & 1+a^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & 1+a^2 \\ 1+a^2 & 1 & a \\ a & 1+a^2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^5+a^4+a^3+a & a^2+a+1 & a^5+a+1 \\ a^5+a+1 & a^5+a^4+a^3+a & a^2+a+1 \\ a^2+a+1 & a^5+a+1 & a^5+a^4+a^3+a \end{pmatrix}. \qquad (2.12)$$

which clearly shows that all the entries in the matrix are non-zero. Therefore, R_2^3 is MDS.

From (2.11) and (2.12), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & a^3+a^2 & & & \\ & & a^4+a^3+a & a^5+a^4+a^3+a & a^4+a^3+a & & \\ & a^4+a^3+a & a^5+a+1 & a^6+a^5+a^4+a & a^2+a+1 & a^4+a^3+a & \\ a^4+a^3+a & a^2+a+1 & a^5+a^3+1 & a^5+a^4+a^3+a & a^3+a^2+1 & a^5+a+1 & a^4+a^3+a \\ & a^3+a^2+1 & a^5+a+1 & a^6+a^5+a^4+a & a^2+a+1 & a^5+a^3+1 & \\ & & a^5+a^3+1 & a^5+a^4+a^3+a & a^3+a^2+1 & & \\ & & & a^6+a^5+a^4+a & & & \end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

∎

Theorem 2.7 Let R_7 be a special type of circulant rhotrix and $R_1 = (a^{-1}, cir(1, a^{-2}, 1+a))$ and $R_2 = cir(1, a^{-2}, a^{-1})$ be defined over $GF(2^8)$, where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_7^3 of order 7.

Proof: Let R_7^3 be defined as in (2.4) and $R_1 = (a^{-1}, cir(1, a^{-2}, 1+a))$ in (2.4), therefore we have

$$R_1^3 = \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-2} & 1+a \\ 1 & 1+a & 1 & a^{-2} \\ 1 & a^{-2} & 1+a & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-2} & 1+a \\ 1 & 1+a & 1 & a^{-2} \\ 1 & a^{-2} & 1+a & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 & 1 & 1 \\ 1 & 1 & a^{-2} & 1+a \\ 1 & 1+a & 1 & a^{-2} \\ 1 & a^{-2} & 1+a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a+a^{-2}+a^{-3} & a^2+a^{-2}+a^{-3}+a^{-4} & a^2+a^{-2}+a^{-3}+a^{-4} & a^2+a^{-2}+a^{-3}+a^{-4} \\ a^2+a^{-2}+a^{-3}+a^{-4} & a^3+a^2+a+a^{-1}+a^{-6} & a^2+a^{-1}+a^{-2}+a^{-3}+a^{-4}+1 & a+a^{-1}+a^{-2}+a^{-4} \\ a^2+a^{-2}+a^{-3}+a^{-4} & a+a^{-1}+a^{-2}+a^{-4} & a^3+a^2+a+a^{-1}+a^{-6} & a^2+a^{-1}+a^{-2}+a^{-3}+a^{-4}+1 \\ a^2+a^{-2}+a^{-3}+a^{-4} & a^2+a^{-1}+a^{-2}+a^{-3}+a^{-4}+1 & a+a^{-1}+a^{-2}+a^{-4} & a^3+a^2+a+a^{-1}+a^{-6} \end{pmatrix} \tag{2.13}$$

This gives,

$R_1^3[1][1] = a + a^{-2} + a^{-3} = a^7 + a^6 + a^5 + a^3 \neq 0$,

$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = a^2 + a^{-2} + a^{-3} + a^{-4} = a^5 + a^4 + a^2 + a \neq 0,$

$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = a^2 + a^{-2} + a^{-3} + a^{-4} = a^5 + a^4 + a^2 + a \neq 0,$

$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = a^3 + a^2 + a + a^{-1} + a^{-6} = a^7 + a^5 + a^4 + a^2 + a \neq 0,$

$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][2] = a^2 + a^{-1} + a^{-2} + a^{-3} + a^{-4} + 1 = a^7 + a^5 + a^4 + a^3 + 1 \neq 0,$

$R_1^3[2][4] = R_1^3[3][2] = R_1^3[4][3] = a^{-4} + a^{-2} + a^{-1} + a = a^4 + a + 1 \neq 0.$

Hence, R_1^3 is MDS for $n = 8$.Now,

$$R_2^3 = \begin{pmatrix} 1 & a^{-2} & a^{-1} \\ a^{-1} & 1 & a^{-2} \\ a^{-2} & a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-2} & a^{-1} \\ a^{-1} & 1 & a^{-2} \\ a^{-2} & a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a^{-2} & a^{-1} \\ a^{-1} & 1 & a^{-2} \\ a^{-2} & a^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-3} + a^{-6} + 1 & a^{-5} & a^{-1} \\ a^{-1} & a^{-3} + a^{-6} + 1 & a^{-5} \\ a^{-5} & a^{-1} & a^{-3} + a^{-6} + 1 \end{pmatrix}. \tag{2.14}$$

The matrix (2.14) gives,

$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = a^{-3} + a^{-6} + 1 = a^7 + a^4 + a^3 + a \neq 0,$

$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][1] = a^{-5} = a^6 + a^5 + a^3 + a^2 \neq 0,$

$R_2^3[1][3] = R_2^3[3][2] = R_2^3[2][1] = a^{-1} = a^7 + a^3 + a^2 + a \neq 0.$

Therefore, R_2^3 is MDS. From (2.13) and (2.14), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc} & & & a^7+a^6+a^5+a^3 & & & \\ & & a^5+a^4+a^2+a & a^7+a^4+a^3+a & a^5+a^4+a^2+a & & \\ & a^5+a^4+a^2+a & a^7+a^3+a^2+a & a^7+a^5+a^4+a^2+a & a^6+a^5+a^3+a^2 & a^5+a^4+a^2+a & \\ a^5+a^4+a^2+a & a^6+a & a^4+a+1 & a^7+a^4+a^3+a & a^7+a^5+a^4+a^3+1 & a^7+a^3+a^2+a & a^5+a^4+a^2+a \\ & a^7+a^5+a^4+a^3+1 & a^7+a^3+a^2+a & a^7+a^5+a^4+a^2+a & a^6+a^5+a^3+a^2 & a^4+a+1 & \\ & & a^4+a+1 & a^7+a^4+a^3+a & a^7+a^5+a^4+a^3+1 & & \\ & & & a^7+a^5+a^4+a^2+a & & & \end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

∎

Theorem 2.8 Let R_7 be a special type of circulant rhotrix and $R_1 = (a, cir(1, 1 + a + a^{-1} + a^{-2}, a + a^{-1}))$ and $R_2 = cir(1, a, a + a^{-1})$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3

form MDS rhotrix R_7^3 of order 7.

Proof: Let R_7^3 be defined as in (2.4) and $R_1 = (a, cir(1, 1 + a + a^{-1} + a^{-2}, a + a^{-1}))$ in (2.4), therefore we have

$$R_1^3 = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} \\ 1 & a+a^{-1} & 1 & 1+a+a^{-1}+a^{-2} \\ 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} \\ 1 & a+a^{-1} & 1 & 1+a+a^{-1}+a^{-2} \\ 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} \\ 1 & a+a^{-1} & 1 & 1+a+a^{-1}+a^{-2} \\ 1 & 1+a+a^{-1}+a^{-2} & a+a^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^{-2}+a^{3} & a^{2}+1+a^{-1}+a^{-4} & a^{2}+1+a^{-1}+a^{-4} & a^{2}+1+a^{-1}+a^{-4} \\ a^{2}+1+a^{-1}+a^{-4} & a^{3}+a+1+a^{-2}+a^{-5}+a^{-6} & a^{3}+a^{2}+1+a^{-1}+a^{-5} & a^{3}+a^{2}+a+a^{-2}+a^{-3}+a^{-4} \\ a^{2}+1+a^{-1}+a^{-4} & a^{3}+a^{2}+a+a^{-2}+a^{-3}+a^{-4} & a^{3}+a+1+a^{-2}+a^{-5}+a^{-6} & a^{3}+a^{2}+1+a^{-1}+a^{-5} \\ a^{2}+1+a^{-1}+a^{-4} & a^{3}+a^{2}+1+a^{-1}+a^{-5} & a^{3}+a^{2}+a+a^{-2}+a^{-3}+a^{-4} & a^{3}+a+1+a^{-2}+a^{-5}+a^{-6} \end{pmatrix}$$

(2.15)

This gives,

$R_1^3[1][1] = a^{-2} + a^{3} = a^{6} + a^{3} + a^{2} + a + 1 \neq 0,,$

$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = a^{2} + 1 + a^{-1} + a^{-4} = a^{6} + a^{4} + a + 1 \neq 0,$

$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = a^{2} + 1 + a^{-1} + a^{-4} = a^{6} + a^{4} + a + 1 \neq 0,$

$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = a^{3} + a + 1 + a^{-2} + a^{-5} + a^{-6} = a^{4} + a^{2} + a \neq 0,$

$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][2] = a^{3} + a^{2} + 1 + a^{-1} + a^{-5} = a^{7} + a^{6} + a^{5} + a^{3} + a^{2} + a + 1 \neq 0,$

$R_1^3[2][4] = R_1^3[3][2] = R_1^3[4][3] = a^{3} + a^{2} + a + a^{-3} + a^{-4} + a^{-2} = a^{5} + a^{3} + a^{2} \neq 0.$

Hence, R_1^3 is MDS for $n = 8$.Now,

$$R_2^3 = \begin{pmatrix} 1 & a & a+a^{-1} \\ a+a^{-1} & 1 & a \\ a & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a & a+a^{-1} \\ a+a^{-1} & 1 & a \\ a & a+a^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & a & a+a^{-1} \\ a+a^{-1} & 1 & a \\ a & a+a^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a+a^{-1}+a^{-3}+1 & a^{3}+a^{2}+a^{-2} & a^{3}+a^{2}+a \\ a^{3}+a^{2}+a & a+a^{-1}+a^{-3}+1 & a^{3}+a^{2}+a^{-2} \\ a^{3}+a^{2}+a^{-2} & a^{3}+a^{2}+a & a+a^{-1}+a^{-3}+1 \end{pmatrix}. \qquad (2.16)$$

The matrix (2.16) gives,

$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = a + a^{-1} + a^{-3} + 1 = a^5 + 1 \neq 0,$

$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][1] = a^3 + a^2 + a^{-2} = a^6 + a^3 + a + 1 \neq 0,$

$R_2^3[1][3] = R_2^3[3][2] = R_2^3[2][1] = a^3 + a^2 + a \neq 0.$

Therefore, R_2^3 is MDS. From (2.15) and (2.16), we obtain

$$R_7^3 = \left\langle \begin{array}{ccccccc}
 & & & a^6+a^3+a^2+a+1 & & & \\
 & & a^6+a^4+a+1 & a^5+1 & a^6+a^4+a+1 & & \\
 & a^6+a^4+a+1 & a^3+a^2+a & a^7+a^5+a^4+a^2+a & a^6+a^3+a+1 & a^6+a^4+a+1 & \\
a^6+a^4+a+1 & a^6+a^3+a+1 & a^5+a^3+a^2 & a^5+1 & a^7+a^6+a^5+a^3+a^2+a+1 & a^3+a^2+a & a^6+a^4+a+1 \\
 & a^7+a^6+a^5+a^3+a^2+a+1 & a^3+a^2+a & a^7+a^5+a^4+a^2+a & a^6+a^3+a+1 & a^5+a^3+a^2 & \\
 & & a^5+a^3+a^2 & a^5+1 & a^7+a^6+a^5+a^3+a^2+a+1 & & \\
 & & & a^7+a^5+a^4+a^2+a & & &
\end{array} \right\rangle,$$

which shows that R_7^3 is MDS.

■

Theorem 2.9 Let R_9 be a special type of circulant rhotrix and $R_1 = (a, cir(1, a, a^{-1}, a^2))$ and $R_2 = cir(1, a, a^{-1}, 1 + a)$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_9^3 of order 9.

Proof: Let R_9^3 be defined as

$$R_9^3 = \left\langle \begin{array}{ccccccccc}
 & & & & R_1^3[1][1] & & & & \\
 & & & R_1^3[2][1] & R_2^3[1][1] & R_1^3[1][2] & & & \\
 & & R_1^3[3][1] & R_2^3[2][1] & R_1^3[2][2] & R_2^3[1][2] & R_1^3[1][3] & & \\
 & R_1^3[4][1] & R_2^3[3][1] & R_1^3[3][2] & R_2^3[2][2] & R_1^3[2][3] & R_2^3[1][3] & R_1^3[1][4] & \\
R_1^3[5][1] & R_2^3[4][1] & R_1^3[4][2] & R_2^3[3][2] & R_1^3[3][3] & R_2^3[2][3] & R_1^3[2][4] & R_2^3[1][4] & R_1^3[1][5] \\
 & R_1^3[5][2] & R_2^3[4][2] & R_1^3[4][3] & R_2^3[3][3] & R_1^3[3][4] & R_2^3[2][4] & R_1^3[2][5] & \\
 & & R_1^3[5][3] & R_2^3[4][3] & R_1^3[4][4] & R_2^3[3][4] & R_1^3[3][5] & & \\
 & & & R_1^3[5][4] & R_2^3[4][4] & R_1^3[4][5] & & & \\
 & & & & R_1^5[5][5] & & & &
\end{array} \right\rangle \qquad (2.17)$$

and $R_1 = (a, cir(1, a, a^{-1}, a^2))$, then

$$R_1^3 = \begin{pmatrix} a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a^2 \\ 1 & a^2 & 1 & a & a^{-1} \\ 1 & a^{-1} & a^2 & 1 & a \\ 1 & a & a^{-1} & a^2 & 1 \end{pmatrix} \begin{pmatrix} a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a^2 \\ 1 & a^2 & 1 & a & a^{-1} \\ 1 & a^{-1} & a^2 & 1 & a \\ 1 & a & a^{-1} & a^2 & 1 \end{pmatrix} \begin{pmatrix} a & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a^2 \\ 1 & a^2 & 1 & a & a^{-1} \\ 1 & a^{-1} & a^2 & 1 & a \\ 1 & a & a^{-1} & a^2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & a^4+a^3+a^2+a^{-2} & a^4+a^3+a^2+a^{-2} & a^4+a^3+a^2+a^{-2} & a^4+a^3+a^2+a^{-2} \\ a^4+a^3+a^2+a+a^{-2} & a^{-2}+1 & a^6+a^4+a^{-1} & a^4+a^2+a+a^{-3}+a^{-1} & a^5+a^3+a^2+a+1 \\ a^4+a^3+a^2+a+a^{-2} & a^5+a^3+a^2+a+1 & a^{-2}+1 & a^6+a^4+a^{-1} & a^4+a^2+a+a^{-3}+a^{-1} \\ a^4+a^3+a^2+a+a^{-2} & a^4+a^2+a+a^{-3}+a^{-1} & a^5+a^3+a^2+a+1 & a^{-2}+1 & a^6+a^4+a^{-1} \\ a^4+a^3+a^2+a+a^{-2} & a^6+a^4+a^{-1} & a^4+a^2+a+a^{-3}+a^{-1} & a^5+a^3+a^2+a+1 & a^{-2}+1 \end{pmatrix}. \tag{2.18}$$

From (2.18), we get

$R_1^3[1][1] = a^3 \neq 0,$,

$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = R_1^3[1][5] = a^4 + a^3 + a^2 + a^{-2} = a^6 + a^4 + a^3 + a + 1 \neq 0,$

$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = R_1^3[5][1] = a^4 + a^3 + a^2 + a + a^{-2} = a^6 + a^4 + a^3 + 1 \neq 0,$

$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = R_1^3[5][5] = a^{-2} + 1 = a^6 + a^2 + a \neq 0,$

$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][5] = R_1^3[5][2] = a^6 + a^4 + a^{-1} = a^7 + a^6 + a^4 + a^3 + a^2 + a \neq 0,$

$R_1^3[2][4] = R_1^3[3][5] = R_1^3[4][2] = R_1^3[5][3] = a^4 + a^2 + a + a^{-3} + a^{-1} = a^5 + a^4 + a^2 + 1 \neq 0,$

$R_1^3[2][5] = R_1^3[3][2] = R_1^3[4][3] = R_1^3[5][4] = a^5 + a^3 + a^2 + a + 1 \neq 0.$

Therefore, R_1^3 is MDS. Now,

$$R_2^3 = \begin{pmatrix} 1 & a & a^{-1} & a+1 \\ a+1 & 1 & a & a^{-1} \\ a^{-1} & a+1 & 1 & a \\ a & a^{-1} & a+1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & a^{-1} & a+1 \\ a+1 & 1 & a & a^{-1} \\ a^{-1} & a+1 & 1 & a \\ a & a^{-1} & a+1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & a^{-1} & a+1 \\ a+1 & 1 & a & a^{-1} \\ a^{-1} & a+1 & 1 & a \\ a & a^{-1} & a+1 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} a^{-2}+a^{-1}+1 & a^{-1}+1 & a^{-3}+a^{-1}+1 & a^{-2}+a^{-1} \\ a^{-2}+a^{-1} & a^{-2}+a^{-1}+1 & a^{-1}+1 & a^{-3}+a^{-1}+1 \\ a^{-3}+a^{-1}+1 & a^{-2}+a^{-1} & a^{-2}+a^{-1}+1 & a^{-1}+1 \\ a^{-1}+1 & a^{-3}+a^{-1}+1 & a^{-2}+a^{-1} & a^{-2}+a^{-1}+1 \end{pmatrix} \quad (2.19)$$

From (2.19), we get

$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = R_2^3[4][4] = a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 \neq 0,$

$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][4] = R_2^3[4][1] = a^{-1} + 1 = a^7 + a^3 + a^2 + a + 1 \neq 0,$

$R_2^3[1][3] = R_2^3[3][1] = R_2^3[2][4] = R_2^3[4][2] = a^{-3} + a^{-1} + 1 = a^5 + a \neq 0,$

$R_2^3[1][4] = R_2^3[2][1] = R_2^3[3][2] = R_2^3[4][3] = a^{-2} + a^{-1} + 1 = a^7 + a^6 + a^3 \neq 0.$

Therefore, R_2^3 is MDS. The matrices (2.18) and (2.19) shows that the rhotrix (2.17) is an MDS rhotrix.

Theorem 2.10 Let R_9 be a special type of circulantrhotrix and $R_1 = (a+1, cir(1, a, a^{-1}, a+1))$ and $R_2 = cir(a, 1, a+1, a^2)$ be defined over GF(2^8), where a is the root of irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$. Then R_1^3 and R_2^3 form MDS rhotrix R_9^3 of order 9.

Proof: Let R_9^3 be defined as in (2.17) and $R_1 = (a+1, cir(1, a, a^{-1}, a+1))$, then

$$R_1^3 = \begin{pmatrix} a+1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a+1 \\ 1 & a+1 & 1 & a & a^{-1} \\ 1 & a^{-1} & a+1 & 1 & a \\ 1 & a & a^{-1} & a+1 & 1 \end{pmatrix} \begin{pmatrix} a+1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a+1 \\ 1 & a+1 & 1 & a & a^{-1} \\ 1 & a^{-1} & a+1 & 1 & a \\ 1 & a & a^{-1} & a+1 & 1 \end{pmatrix} \begin{pmatrix} a+1 & 1 & 1 & 1 & 1 \\ 1 & 1 & a & a^{-1} & a+1 \\ 1 & a+1 & 1 & a & a^{-1} \\ 1 & a^{-1} & a+1 & 1 & a \\ 1 & a & a^{-1} & a+1 & 1 \end{pmatrix}$$

$$
=\begin{pmatrix}
(a+1)^3 & a^2+a^{-2}+a^{-1} & a^2+a^{-2}+a^{-1} & a^2+a^{-2}+a^{-1} & a^2+a^{-2}+a^{-1} \\
a^2+a^{-2}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-1} & a+a^{-3}+a^{-1} & a+a^{-2}+a^{-1} \\
a^2+a^{-2}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-1} & a+a^{-3}+a^{-1} \\
a^2+a^{-2}+a^{-1} & a+a^{-3}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-1} \\
a^2+a^{-2}+a^{-1} & a+a^{-1} & a+a^{-3}+a^{-1} & a+a^{-2}+a^{-1} & a+a^{-2}+a^{-1}
\end{pmatrix}. \tag{2.20}
$$

From (2.20), we get

$R_1^3[1][1] = a^3 + a^2 + a + 1 \neq 0,$

$R_1^3[1][2] = R_1^3[1][3] = R_1^3[1][4] = R_1^3[1][5] = a^2 + a^{-2} + a^{-1} = a^7 + a^6 + a^3 + a^2 + 1 \neq 0,$

$R_1^3[2][1] = R_1^3[3][1] = R_1^3[4][1] = R_1^3[5][1] = a^2 + a^{-2} + a^{-1} = a^7 + a^6 + a^3 + a^2 + 1 \neq 0,$

$R_1^3[2][2] = R_1^3[3][3] = R_1^3[4][4] = R_1^3[5][5] = a^{-2} + a^{-1} + a = a^7 + a^6 + a^3 + a + 1 \neq 0,$

$R_1^3[2][3] = R_1^3[3][4] = R_1^3[4][5] = R_1^3[5][2] = a + a^{-1} = a^7 + a^3 + a^2 \neq 0,$
$R_1^3[2][4] = R_1^3[3][5] = R_1^3[4][2] = R_1^3[5][3] = a + a^{-3} + a^{-1} = a^5 + 1 \neq 0,$
$R_1^3[2][5] = R_1^3[3][2] = R_1^3[4][3] = R_1^3[5][4] = a^{-2} + a^{-1} + a = a^7 + a^6 + a^3 + a + 1 \neq 0.$

Therefore, R_1^3 is MDS. Now,

$$
R_2^3 = \begin{pmatrix} a & 1 & a+1 & a^2 \\ a^2 & a & 1 & a+1 \\ a+1 & a^2 & a & 1 \\ 1 & a+1 & a^2 & a \end{pmatrix}\begin{pmatrix} a & 1 & a+1 & a^2 \\ a^2 & a & 1 & a+1 \\ a+1 & a^2 & a & 1 \\ 1 & a+1 & a^2 & a \end{pmatrix}\begin{pmatrix} a & 1 & a+1 & a^2 \\ a^2 & a & 1 & a+1 \\ a+1 & a^2 & a & 1 \\ 1 & a+1 & a^2 & a \end{pmatrix}
$$

$$
=\begin{pmatrix}
a^5+a^4+1 & a^6+a^2+1 & a^5+1 & a^4+a^2+1 \\
a^4+a^2+1 & a^5+a^4+1 & a^6+a^2+1 & a^5+1 \\
a^5+1 & a^4+a^2+1 & a^5+a^4+1 & a^6+a^2+1 \\
a^6+a^2+1 & a^5+1 & a^4+a^2+1 & a^5+a^4+1
\end{pmatrix}. \tag{2.21}
$$

From (2.21), we get

$R_2^3[1][1] = R_2^3[2][2] = R_2^3[3][3] = R_2^3[4][4] = a^5 + a^4 + 1 \neq 0,$

$$R_2^3[1][2] = R_2^3[2][3] = R_2^3[3][4] = R_2^3[4][1] = a^6 + a^2 + 1 \neq 0,$$

$$R_2^3[1][3] = R_2^3[3][1] = R_2^3[2][4] = R_2^3[4][2] = a^5 + 1 \neq 0,$$

$$R_2^3[1][4] = R_2^3[2][1] = R_2^3[3][2] = R_2^3[4][3] = a^4 + a^2 + 1 \neq 0.$$

Therefore, R_2^3 is MDS. The matrices (2.20) and (2.21) shows that the rhotrix (2.17) is an MDS rhotrix.

3. CONCLUSION

We introduced circulant and special form of circulant rhotrices.We constructed MDS rhotrices using special form of circulant rhotrices with entries using the elements $a, a+1, a^2, a^{-1}$ where a is the root of the constructing irreducible polynomial $a^8 + a^4 + a^3 + a^2 + 1$ over GF(2^8).

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HILL CIPHER CRYPTOSYSTEM USING IRREDUCIBLE POLYNOMIALS OVER FINITE FIELDS

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ABSTRACT

Cryptography has a significant role in the present scenario of information. It provides security and integrity to the messages which travel over the insecure channels, and authenticity to the communicating parties. Hill cipher is a symmetric cryptosystem that is used to protect information or data. We propose an algorithm which enhance the security of the Hill cipher by using matrices and irreducible polynomials over finite fields.

AMS classification: 11T71, 94A60**.**

Keywords: Plain text; Cipher text; Irreducible polynomial; Finite field.

1 INTRODUCTION

Information security has become a very critical aspect of modern computing systems. Cryptography is the science which provides confidentiality, authenticity and integrity of information passing through insecure channels,see [10,11]. Although the ultimate goal of cryptography is to hide information from unauthorized individuals. Most algorithms can be broken and the information can be revealed if the attacker has enough time, desire, and resources. As a result researchers are using new techniques from different areas of mathematics like matrix analysis, finite fields [6, 23-26] etc. for the security of data during transmission. There are similar structures to matrices which are known as rhotrices. Such structures came into existence in the literature since 2003. Various researchers have used these rhotrices to develop their structures and apply the same in the field of cryptography, see [12-19].

Hill Cipher is an application of linear algebra to cryptography. The Hill Cipher is classical symmetric cipher invented by Lester S. Hill in 1929 [3] and extension of this work is in [4]. The main advantages of Hill cipher includes its frequency analysis, high speed, high throughput and the simplicity because it uses matrix operations but it succumbs to the known plaintext attack [5]. Hill cipher is modified by several authors. Saeednia [7] uses the dynamic key matrix while Chefranov [2] uses a pseudo-random permutation generator. Ismail et al. [5] give an initial vector to form a different key for each block encryption. Adi et al. [1] modify the Hill cipher using circulant matrices. Shastry et al. [8, 9] use the key on both sides of the plain text to modify Hill cipher.

Sharma and Rehan [20, 21] modify Hill cipher using logical operator. Sharma and Sharma [22] modify Hill cipher using elements of finite field. We give an algorithm along with illustration which involves the encryption and decryption of plaintext by using irreducible polynomials over finite field $GF(2)$. In the proposed cipher, we use the following matrices and the irreducible polynomial.

Vandermonde matrix : A matrix $V(a_1, a_2, \dots, a_m)$ of order $m \times n$ having terms in each row with a geometric progression is called Vandermonde matrix and is written as

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \dots & a_m^{n-1} \end{bmatrix}.$$

Coefficient matrix: Let A be a $n \times n$ matrix, then the coefficient matrix is defined as $circ\big(circ(row\ 1), circ(row\ 2), \dots, circ(rown)\big)$, where$row\ 1, row\ 2, \dots., rown$ are rows of matrix A and $circ(row\ 1)$is the circulant matrix of row 1. It is denoted by A_c.

Example: If A be a2×2 matrix, then its coefficient matrix A_c is 4×4.

$$A = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix},$$

$$A_c = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 & g_1 & g_4 & g_3 \\ g_3 & g_4 & g_1 & g_2 \\ g_4 & g_3 & g_2 & g_1 \end{bmatrix}.$$

Representation of elements in finite fields:

There are number of different representations for the elements of finite fields such as polynomial, binary and decimal.

Example: The following table shows different representations for the elements of $GF(2^3)$ with respect to the irreducible polynomial $f(x) = x^3 + x + 1$. Let the element α be a root of the irreducible polynomial $f(x)$, therefore, $f(\alpha) = 0$. This gives, $\alpha^3 = \alpha + 1$, which is used to reduce the higher power of α.

Powers of α	Polynomial representation	Binary	Decimal
0	0	0 0 0	0
α^0	1	0 0 1	1
α^1	α	0 1 0	2
α^2	α^2	1 0 0	4
α^3	$\alpha + 1$	0 1 1	3
α^4	$\alpha^2 + \alpha$	1 1 0	6
α^5	$\alpha^2 + \alpha + 1$	1 1 1	7
α^6	$\alpha^2 + 1$	1 0 1	5

Table-I

2 ALGORITHM OF THE PROPOSED CIPHER

In the proposed algorithm, we use the elements of finite field in binary, polynomial form and also use irreducible polynomials over Galois field $GF(2^m)$.

ENCRYPTION:

1. Sender consider a $n \times n$Vander monde matrix S as secret key.
2. He chooses a $n \times n$ non singular matrix A as public key such that det$(A_c) = 0$.
3. Sender calculates key $K_1 = SAS^{-1}(mod\,p)$, where p is an irreducible polynomial of degree m over finite field $GF(2)$.
4. The sender converts the plaintext into numerical values by using Table –II(given on page 4).
5. He then converts the numerical values into binary strings of m-bits.
6. Further, he converts m-bits binary strings into polynomial form.
7. Sender calculates $S_1 = K_1M(mod\,p)$. Each entry of S_1 is multiplied wit$h\, x^m$ and sender calculates K_2,whose entries are 0 if x has the power less than$2^m - 1$otherwise 1 and shares it with the receiver.
8. He then reduces the powers of the entries to $mod\ (2^m - 1)$ and gets the matrix S_3.
9. After writing it into binary form, he converts the same in numerical values and then in text to get the final cipher text S_4.

DECRYPTION :

1. Receiver receives the message. He convert the message into numerical values by using the Table – II(given on page 4).
2. Receiver then converts the numerical values into binary string of m -bits.
3. Then receiver converts the binary strings into the elements of $GF(2^m)$ to get S_3.
4. He then multiplies each entry of S_3 with x^{2^m-1} which represents 1 in the matrix K_2.
5. Receiver multiplies each entry with x^{-m} to obtain S_1.
6. Sender calculate key $K_1^{-1} = SA^{-1}S^{-1}(mod\ p)$, where p is an irreducible polynomial of degree m over finite field $GF(2)$.
7. H calculates $M = K_1^{-1}S_1(modp)$.
8. Then he converts the entries into binary strings to get P_1.
9. Then the receiver converts the entries of P_1 into numerical values. After writing it into numerical values, he converts the same into text to get plaintext.

Numerical values for alphabets and some symbols used in the paper

[	]	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
O	P	Q	R	S	T	U	V	W	X	Y	Z				
17	18	19	20	21	22	23	24	25	26	27	28				

Table-II

3 ILLUSTRATION OF THE CIPHER

Let us consider the following plain text which is to be sent over an insecure channel is **[CD].** Further, we consider the irreducible polynomial $x^3 + x + 1$ with α as its root and finite field $GF(2^3)$.

ENCRYPTION:

Step 1. Sender considers the 2× 2 Vandermonde matrix S.

$$S= \begin{bmatrix} 1 & x^2+1 \\ 1 & x^2+x \end{bmatrix}, \text{ where } x^2+1, x^2+x \in GF(2^3).$$

Step 2. Select a 2× 2 non singular matrix A whose elements are from $GF(2^3)$ as public key.

$$A= \begin{bmatrix} x & x+1 \\ x^2 & x^2+1 \end{bmatrix}.$$

Step 3. Calculate the key

$K_1 = \mathrm{SA}S^{-1} \;=\; \begin{bmatrix} 1 & x^2+1 \\ 1 & x^2+x \end{bmatrix} \begin{bmatrix} x & x+1 \\ x^2 & x^2+1 \end{bmatrix} \begin{bmatrix} x & x^3 \\ x^2+x & x^2+x \end{bmatrix} (mod x^3+x+1).$

$= \begin{bmatrix} x^2+1 & x^2+1 \\ x^2+1 & x \end{bmatrix}.$

Step 4. Sender converts the plaintext **[CD]**into numerical values using Table -II as follows

$$P = \begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}.$$

Step 5. He converts the above numerical values into binary string 3-bits and therefore P becomes

$$P_1 = \begin{bmatrix} 001 & 101 \\ 110 & 010 \end{bmatrix}.$$

Step 6. Sender further converts the 3-bits binary string into polynomial form and therefore P_1 gives

$$M = \begin{bmatrix} 1 & x^2+1 \\ x^2+x & x \end{bmatrix}.$$

Step 7. He calculates$S_1 = K_1\, M = \begin{bmatrix} x^2+1 & x^2+1 \\ x^2+1 & x \end{bmatrix} \begin{bmatrix} 1 & x^2+1 \\ x^2+x & x \end{bmatrix} (mod x^3+x+1)$

$= \begin{bmatrix} x^2+x & x^2+x \\ x & x+1 \end{bmatrix}$

Using Table-I, we get

$$S_1 = \begin{bmatrix} x^4 & x^4 \\ x^8 & x^3 \end{bmatrix}.$$

In order to make the exponent of maximum entries of S_1as 7 = (2^3-1), we multiply each entry by x^3. Therefore, S_1becomes

$$S_2 = \begin{bmatrix} x^7 & x^7 \\ x^{11} & x^6 \end{bmatrix}.$$

and the key matrix

$$K_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is chosen in such a way that if power of x in S_2 is less than 7, the entry in the key matrix is taken 0 otherwise 1.

Step 8. The powers of elements of S_2 are reduced by mod 7 and hence it becomes

$$S_3 = \begin{bmatrix} x^0 & x^0 \\ x^4 & x^6 \end{bmatrix}.$$

Step 9. The elements of cipher text matrix S_3 are converted into the binary elements as follows

$$S_4 = \begin{bmatrix} 001 & 001 \\ 110 & 101 \end{bmatrix}.$$

The entries of S_4 are converted into numerical values as follows

$$S_5 = \begin{bmatrix} 1 & 1 \\ 6 & 5 \end{bmatrix}.$$

Further, numerical values are converted into Cipher text =**[[DC.**

The cipher text is sent to the receiver through public channel.

DECRYPTION:

Step 1. Receiver receives the message. He converts the message into numerical values by using Table -II, which gives

$$S_5 = \begin{bmatrix} 1 & 1 \\ 6 & 5 \end{bmatrix}.$$

Step 2. He converts the numerical values into binary strings of 3-bits as follows

$$S_4 = \begin{bmatrix} 001 & 001 \\ 110 & 101 \end{bmatrix}.$$

Step 3. Further, he converts binary strings into the elements of $GF(2^3)$, so S_4 becomes

$$S_3 = \begin{bmatrix} x^0 & x^0 \\ x^4 & x^6 \end{bmatrix}.$$

Step 4. Receiver multiplies only those entries of S_3 by x^7, which represents 1 in the matrix K_2.

$$S_2 = \begin{bmatrix} x^7 & x^7 \\ x^{11} & x^6 \end{bmatrix}.$$

Step 5. He multiplies each entry of S_2 with x^{-3} and obtain

$$S_1 = \begin{bmatrix} x^4 & x^4 \\ x^8 & x^3 \end{bmatrix}$$

Further, $S_1(mod x^3 + x + 1)$ can be written as

$$S_1 = \begin{bmatrix} x^2 + x & x^2 + x \\ x & x + 1 \end{bmatrix}.$$

Step 6. Calculate the key $K_1^{-1} = SA^{-1}S^{-1}$ =

$$\begin{bmatrix}1 & x^2+1\\ 1 & x^2+x\end{bmatrix}\begin{bmatrix}x^2 & x^2+1\\ x^2+x+1 & x^2+1\end{bmatrix}\begin{bmatrix}x & x^3\\ x^2+x & x^2+x\end{bmatrix}(mod\ \ ^3+x+1)$$

$$= \begin{bmatrix}x^2+x & x^2\\ x^2 & x^2\end{bmatrix}.$$

Step 7. The receiver calculates $K_1^{-1}S_1(modx^3+x+1)$.

$$K_1^{-1}S_1 = \begin{bmatrix}x^2+x & x^2\\ x^2 & x^2\end{bmatrix}\begin{bmatrix}x^2+x & x^2+x\\ x & x+1\end{bmatrix}(modx^3+x+1)$$

$$= \begin{bmatrix}1 & x^2+1\\ x^2+x & x\end{bmatrix}.$$

Step 8. Receiver converts the message into binary strings, which gives

$$P_1 = \begin{bmatrix}001 & 101\\ 110 & 010\end{bmatrix}.$$

Step 9. He converts the binary strings into numerical values as follows

$$P = \begin{bmatrix}1 & 5\\ 6 & 2\end{bmatrix}.$$

Receiver converts the digits in text by using Table -II and the plain text **[CD]**is obtained.

4 CONCLUSION

The proposed cipher is the enhanced form of the Hill cipher. With the addition of Vandermonde matrix and modulo irreducible polynomials, the cipher has increased its security. The introduced mechanism in the cipher has created difficulty to the hackers to break the system and retrieve the original message from the cipher text.

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On Exchange Principle in Rotatory Hydrodynamic Triply Diffusive Convection in Porous Medium: Darcy Model

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Abstract

The present paper mathematically establishes that 'the principle of the exchange of stabilities' for rotatory hydrodynamic triply diffusive convection in porous medium is valid in the regime $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+TP_1^2 \leq 1,$ where R_1 and R_2 are the Rayleigh numbers for the two concentration components, T is the Taylor number, P_1 is a constant, τ_1 and τ_2 are the Lewis numbers for the two concentration components and σ is the thermal Prandtl number. It is further proved that the above result is uniformly valid for any combination of rigid and free boundaries.

Keywords: Triply diffusive convection; Principle of the exchange of stabilities; Concentration Rayleigh number; Taylor number; Porous medium; Darcy Model

1. Introduction

Research on convective fluid motion in porous media under the simultaneous action of a uniform vertical temperature gradient and a gravitationally opposite uniform vertical concentration gradient (known as double diffusive convection) has been an area of great activity due to its importance in the predication of ground water movement in aquifers, in assessing the effectiveness of fibrous materials, in engineering geology and in nuclear engineering. Most of the researchers have studied double diffusive convection in porous medium by considering the Darcy flow model which is relevant to densely packed, low permeability porous medium. Double diffusive convection is now well known. For a broad view of the subject one may be referred to Nield and Bezan [10], Murray and Chen [9], Nield [11], Taunton et al. [21], Kuznetsov and Nield [6], Vafai [26] and Kellner and Tilgner [5].

All these researchers have considered double diffusive convection. However, it has been recognized later that there are many fluid systems, in which more than two components are present. The oceans contain many salts having concentrations less than a few percent of the sodium chloride concentration. Multi-component concentrations can also be found in magmas and substratum of water reservoirs. The subject with more than two components (in porous and non porous medium) has attracted the attention of many researchers Griffiths [2, 3], Poulikakos [14], Pearlstein et al. [12], Terrones and Pearlstein [22], Rudraiah and Vortmeyer [16], Lopez et al. [7], Tracey [23, 24], Straughan and Tracey [19], and Rionero [15]. The essence of the works of these researchers is that small salinity of a third component with a smaller mass diffusivity can have a significant effect upon the nature of convection; and 'oscillatory' and direct 'salt finger' modes are simultaneous possible under a wide range of conditions, when the density gradients due to components with greatest and smallest diffusivity are of same signs.

Double or triply diffusive convection in a rotating fluid layer saturating a porous medium is an interesting topic due to its applications in chemical process industry, food processing industry, solidification and centrifugal costing of metals and rotating machinery, petroleum industry, geophysics and biomechanics. Several studies are available in which phenomena related to the onset of single diffusive (Benard Problem) and double diffusive convection in a rotating porous medium have been investigated. For a detailed review of the subject one may be referred to Vadasz [25], Nield and Bezan [10], Tagare et al. [20], Sengupta and Gupta [18], Malashetty and Begum [8]. To the authors knowledge no such significant work has been done so far in rotatory hydrodynamic triply diffusive convection in porous medium.

The establishment of the non occurrence of any slow oscillatory motions which may be neutral or unstable implies the validity of the principle of the exchange of stabilities (PES). The validity of this principle in stability problems eliminates the unsteady terms from the linearized perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is characterized by the vanishing of both real and imaginary parts of the complex time eigen value associated with the perturbation. Pellew and Southwell [13] proved the validity of PES (i.e. occurrence of stationary convection) for the classical Rayleigh-Benard instability problem. However no such results existed for other more general hydrodynamic configurations. Banerjee et al. [1] established such a criterion for magnetohydrodynamic Rayleigh-Benard convection problem which has further been extended by Gupta et al. [4] for thermohaline convection problems.

The aim of the present paper is to establish criteria for characterizing non oscillatory motions which may be neutral or unstable for rotatory hydrodynamic triply diffusive configuration in porous medium. It is proved that for rotatory hydrodynamic triply diffusive convection in porous medium, if $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+TP_1^2 \leq 1$, then an arbitrary neutral or unstable mode of the system is definitely non oscillatory in character and in particular PES is valid where R_1 and R_2 are concentration Rayleigh numbers for the two concentration components, T is the Taylor number, τ_1 and τ_2 are the Lewis numbers for the two concentrations, σ is the Prandtl number. It is further proved that the above result is uniformly valid for all the combinations of rigid and free boundaries.

2. Mathematical Formulation of the Problem

An infinite horizontal porous layer filled with a viscous and Boussinesq fluid, statically confined between two horizontal boundaries $z = 0$ and $z = d$, respectively maintained at uniform constant temperatures T_0 and T_1 $(< T_0)$ and uniform concentrations S_{10} , S_{20} and $S_{11}(< S_{10})$, $S_{21}(< S_{20})$ is kept rotating at a constant rate $\vec{\Omega}$ about the vertical (as shown in fig. 1). It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Darcy model has been used to investigate the triply diffusive convection in porous medium.

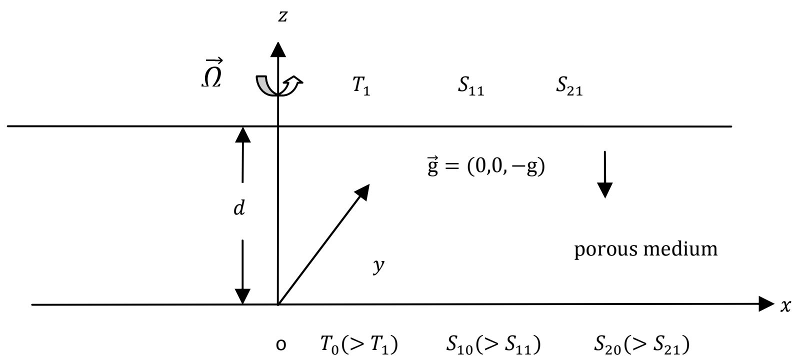

Fig. 1. Physical Configuration

The equations that govern the motion of triply diffusive fluid layer in a porous medium (Darcy Model) under the action of a uniform vertical rotation, in the non-dimensional form, are as follows (Wankat and Schowalter [27], Vafai [26]):

$$\left(\frac{p}{\sigma}+\frac{1}{P_1}\right)(D^2-a^2)w = -Ra^2\theta + R_1a^2\phi_1 + R_2a^2\phi_2 - TD\zeta, \tag{1}$$

$$(D^2-a^2-Ap)\theta = -w, \tag{2}$$

$$\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{3}$$

$$\left(D^2-a^2-\frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \tag{4}$$

and $\left(\frac{p}{\sigma}+\frac{1}{P_1}\right)\zeta = Dw.$ (5)

Eqs. (1) - (5) are to be solved using the following boundary conditions:

$w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta$ at $z = 0$ and $z = 1$, (6)

(Both the boundaries are dynamically free)

Or $w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta$ at $z = 0$ and $z = 1$, (7)

(Both the boundaries are rigid)

$w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta$ at $z = 0$, (8)

(lower boundary is dynamically free)

and $w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta$ at $z = 1$, (9)

(upper boundary is rigid)

Eqs. (1) – (5) together with the boundary conditions (6) – (9) present an eigen value problem for p for the given values of the other parameters and govern rotatory triply diffusive convection in a porous medium.

The meaning of the symbols involved in Eqs. (1)-(9) from the physical point of view are as follows : z is the vertical coordinate, D is the differentiation w.r.t. z , a^2 is square of the wave number, $\sigma > 0$ is the Prandtl number, $\tau_1 > 0$ and $\tau_2 > 0$ are the Lewis numbers for the two concentration components with mass diffusivity κ_1, κ_2 respectively, $R > 0$ is the thermal Rayleigh number, $R_1 > 0$ and $R_2 > 0$ are the two concentration Rayleigh numbers, $T > 0$ is Taylor number, $p = p_r + i\, p_i$ is the complex growth rate where p_r and p_i are real constants, w is the vertical velocity, θ is the temperature and ϕ_1 and ϕ_2 are the two concentrations. The governing equations also involve two more positive constants namely $P_1 = \frac{k_1}{\epsilon d^2}$ and $A = 1 + \frac{\rho_{s_0} c_{s_0}}{\rho_0 c_0}\frac{1-\epsilon}{\epsilon}$, where k_1 is the permeability, ϵ is the porosity of the medium, d is the depth of the fluid layer, $\rho_{s_0}$is the solid density, c_{s_0} is the heat capacity of the solid. The suffix '0' denotes the values of various parameters involved in the governing equations at some properly chosen mean temperature T_0 .

We prove the following theorem:

Theorem: If $(w, \theta, \phi_1, \phi_2, p, \zeta)$, $p_r \geq 0$ is a solution of equations (1) – (9) with $R > 0$, $R_1 > 0$, $R_2 > 0, T > 0$ and $\frac{R_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 \sigma}{2\tau_2^2 \pi^4} + TP_1^2 \leq 1$, then $p_i = 0$. In particular $p_r = 0$ implies $p_i = 0$, if $\frac{R_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 \sigma}{2\tau_2^2 \pi^4} + TP_1^2 \leq 1$.

Proof: Multiplying Eq. (1) by w^* (the complex conjugate of w) and integrating the resulting equation over the vertical range of *z,* we obtain

$$\left(\frac{p}{\sigma} + \frac{1}{P_1}\right) \int_0^1 w^* (D^2 - a^2) w dz = -\, R a^2 \int_0^1 w^* \theta\, dz + R_1 a^2 \int_0^1 w^* \phi_1\, dz + R_2 a^2 \int_0^1 w^* \phi_2\, dz - T \int_0^1 w^* D\zeta\, dz\,. \tag{10}$$

Making use of Eqs. (2) – (5) and the fact that $w(0) = 0 = w(1)$, we can write

$$-Ra^2 \int_0^1 w^* \theta\, dz = \mathrm{R}\, a^2 \int_0^1 \theta\, (D^2 - a^2 - Ap^*) \theta^*\, dz, \tag{11}$$

$$R_1 a^2 \int_0^1 w^* \phi_1\, dz = -\, R_1 a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right) \phi_1{}^*\, dz, \tag{12}$$

$$R_2 a^2 \int_0^1 w^* \phi_2\, dz = -R_2 a^2 \tau_2 \int_0^1 \phi_2\, (D^2 - a^2 - \frac{p^*}{\tau_2}) \phi_2{}^*\, dz, \tag{13}$$

$$-T \int_0^1 w^* D\zeta\, dz = T \left(\frac{p^*}{\sigma} + \frac{1}{P_1}\right) \int_0^1 |\zeta|^2\, dz. \tag{14}$$

Combining Eqs. (10) – (14), we get

$$\left(\frac{p}{\sigma}+\frac{1}{P_1}\right)\int_0^1 w^*(D^2-a^2)w dz = \mathrm{R}\, a^2\int_0^1 \theta\,(D^2-a^2-Ap^*)\theta^*\,dz - R_1 a^2\tau_1$$
$$\int_0^1 \phi_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)\phi_1{}^*\,dz - R_2 a^2\tau_2\int_0^1 \phi_2\,(D^2-a^2-\frac{p^*}{\tau_2})\phi_2{}^*\,dz$$
$$+T\left(\frac{p^*}{\sigma}+\frac{1}{P_1}\right)\int_0^1 |\zeta|^2\,dz. \quad (15)$$

Integrating the various terms of Eq. (15), by parts, for an appropriate number of times and utilizing the boundary conditions (6) - (9), we obtain

$$\left(\frac{p}{\sigma}+\frac{1}{P_1}\right)\int_0^1 (|Dw|^2+a^2|w|^2)dz = R\,a^2\int_0^1 (|D\theta|^2+a^2|\theta|^2+Ap^*|\theta|^2)dz -$$
$$R_1 a^2\tau_1\int_0^1\left(|D\phi_1|^2+a^2|\phi_1|^2+\frac{p^*}{\tau_1}|\phi_1|^2\right)dz - R_2 a^2\tau_2\int_0^1\left(|D\phi_2|^2+a^2|\phi_2|^2+\right.$$
$$\left.\frac{p^*}{\tau_2}|\phi_2|^2\right)dz - T\left(\frac{p^*}{\sigma}+\frac{1}{P_1}\right)\int_0^1 |\zeta|^2\,dz. \quad (16)$$

Equating the imaginary parts of both sides of Eq. (16) and cancelling $p_i\,(\neq 0)$ throughout from the resulting equation, we have

$$\frac{1}{\sigma}\int_0^1 (|Dw|^2+a^2|w|^2)dz = -RAa^2\int_0^1 |\theta|^2 dz + R_1 a^2\int_0^1 |\phi_1|^2 dz + R_2 a^2\int_0^1 |\phi_2|^2 dz$$
$$+\frac{T}{\sigma}\int_0^1 |\zeta|^2\,dz. \quad (17)$$

Now, multiplying equation (3) by its complex conjugate, we obtain

$$\int_0^1\left(D^2-a^2-\frac{p}{\tau_1}\right)\phi_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)\phi_1^* dz = \frac{1}{\tau_1^2}\int_0^1 |w|^2 dz. \quad (18)$$

Integrating the various terms on the left hand side of equation (18), by parts, for an appropriate number of times and making use of the boundary conditions on ϕ_1, it follows that

$$\int_0^1 (|D^2\phi_1|^2+2a^2|D\phi_1|^2+a^4|\phi_1|^2)dz + \frac{2p_r}{\tau_1}\int_0^1 (|D\phi_1|^2+a^2|\phi_1|^2)dz +$$
$$\frac{|p|^2}{\tau_1^2}\int_0^1 |\phi_1|^2\,dz = \frac{1}{\tau_1^2}\int_0^1 |w|^2\,dz. \quad (19)$$

Since $p_r \geq 0$, it follows from equation (19), that

$$2a^2\int_0^1 |D\phi_1|^2 dz < \frac{1}{\tau_1^2}\int_0^1 |w|^2\,dz. \quad (20)$$

Now, since ϕ_1, ϕ_2 and w satisfy the boundary conditions, namely, $\phi_1(0)=0=\phi_1(1)$, $\phi_2(0)=0=\phi_2(1), w(0)=0=w(1)$ respectively, we have by Rayleigh-Ritz inequality (Schultz [17])

$$\int_0^1 |D\phi_1|^2 dz \geq \pi^2 \int_0^1 |\phi_1|^2 dz, \tag{21}$$

$$\int_0^1 |D\phi_2|^2 dz \geq \pi^2 \int_0^1 |\phi_2|^2 dz, \tag{22}$$

and

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \tag{23}$$

Utilizing inequalities (21) and (23) in inequality (20), we get

$$a^2 \int_0^1 |\phi_1|^2 dz \leq \frac{1}{2\tau_1^2 \pi^4} \int_0^1 |Dw|^2 dz. \tag{24}$$

In the same manner, we obtain from equation (4), the inequality

$$a^2 \int_0^1 |\phi_2|^2 dz \leq \frac{1}{2\tau_2^2 \pi^4} \int_0^1 |Dw|^2 dz. \tag{25}$$

Multiplying Eq. (5) by ζ^* on both sides and equating real parts on both sides, we obtain

$$\frac{p_r}{\sigma} \int_0^1 |\zeta|^2 \, dz + \frac{1}{P_1} \int_0^1 |\zeta|^2 \, dz = \text{real part of} \left(\int_0^1 \zeta^* Dw \, dz \right)$$

$$\leq \left| \int_0^1 \zeta^* Dw dz \right| \leq \int_0^1 |\zeta^* Dw| dz$$

$$\leq \left(\int_0^1 |Dw|^2 dz \right)^{1/2} \left(\int_0^1 |\zeta|^2 dz \right)^{1/2}, \qquad \text{(using Schwartz inequality)}$$

which implies that

$$\frac{1}{P_1} \left(\int_0^1 |\zeta|^2 dz \right)^{1/2} \leq \left(\int_0^1 |Dw|^2 dz \right)^{1/2}, \tag{26}$$

which implies

$$\int_0^1 |\zeta|^2 dz \leq P_1^2 \int_0^1 |Dw|^2 dz. \tag{27}$$

Using inequalities (24), (25) and (27) in equation (17), we get

$$\left[\frac{1}{\sigma} - \left(\frac{R_1}{2\tau_1^2 \pi^4} + \frac{R_2}{2\tau_2^2 \pi^4} + \frac{T P_1^2}{\sigma} \right) \right] \int_0^1 |Dw|^2 dz + \frac{a^2}{\sigma} \int_0^1 |w|^2 dz + RA \, a^2 \int_0^1 |\theta|^2 dz < 0, \tag{28}$$

which clearly implies (for $p_i \neq 0$) that

$$\frac{R_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 \sigma}{2\tau_2^2 \pi^4} + T P_1^2 > 1.$$
(29)

Hence if $\frac{R_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 \sigma}{2\tau_2^2 \pi^4} + T P_1^2 \leq 1$, then we must have $p_i = 0$.

This establishes the desired result.

The essential content of the theorem from the physical point of view is that for the problem of rotatory hydrodynamic triply diffusive convection in a porous medium of an arbitrary neutral or unstable mode of the system is definitely non oscillatory in character if $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+TP_1^2 \leq 1$ and in particular PES is valid if $\frac{R_1\sigma}{2\tau_1^2\pi^4}+\frac{R_2\sigma}{2\tau_2^2\pi^4}+TP_1^2 \leq 1$.

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Stability of stratified viscoelastic Walters' (model B') fluid/plasma in hydromagnetics in the presence of quantum physics

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ABSTRACT

Quantum effects on the Rayleigh-Taylor Instability in an inhomogeneous stratified incompressible, viscoelastic Walters' (model B') fluid/plasma in hydromagnetics are investigated. The linear growth rate is derived for the case where a plasma with exponential density, viscosity, viscoelasticity and quantum parameter distribution is confined between two rigid planes at $z = 0, z = d$. The solution of the linearized equations of the system together with the boundary conditions leads to derive the dispersion relation (the relation between the normalized growth rate and square normalized behaviour wave number) using normal mode technique to explain the roles that play the variables of the problem. The behaviour of growth rates with respect to the quantum effect and kinematic viscoelasticity are examined in the presence of kinematic viscosity. The results show that the vertical magnetic field brings about more stability for a certain wave number band on the growth rate of unstable configuration.

1. Introduction

The Rayleigh-Taylor instability is an important hydrodynamic effect that arises when a heavy fluid is accelerated into a lighter one. Similar to pouring water into oil, the heavier fluid, once perturbed, streams to the bottom, pushing the light fluid aside. Chandrasekhar (1961) has given a detailed account of these investigations. A good account of such hydrodynamic stability problems has also been given by Drazin and Reid (1981) and Joseph (1976). This class of fluids is mainly used to analyze the frequency of gravity waves in deep oceans, liquid vapour/globe, to extract oil from the earth to eliminate water drops, laser etc. Quantum plasma can be composed of electrons, ions, positrons, holes and or grains.

Quantum plasmas play an important role in ultra small electronic devices which has been given by Dutta and McLennan (1990), dense astrophysical plasmas system has been given by Madappa et al. (2001), intense laser-matter experiments has been investigated by Remington et al. (1999) and non-linear quantum optics has been given by Brambilla et al. (1995). It is well known that quantum effects become important in the behaviour of the charged plasma particles when the de-Broglie wavelength of the charged carriers become equal to or greater than the dimension of the quantum plasma system has been investigated by Kaushal (2001). It should be observed that there is a difference between a light-wave and the de Broglie or Schrodinger wave associated with the light-quanta. Firstly, the light-wave is always real, while the de Broglie wave associated with a light-quantum moving in a definite direction must be taken to involve an imaginary exponential.

While naturally occurring plasma is relatively unusual on earth, it is playing a larger and increasingly important role in how we use and develop modern technology. For instance, producing compact chips on an industrial scale is only made possible by the application of plasma. Plasma is also a key technology in the development of alternative energy sources. Nuclear fusion, which is plasma based, is one of the most promising candidates for the energy needs of the future when fossil fuels finally run out. Plasma is increasingly becoming part of the industrial area and its range of application is vast. The different variables of plasma play important role in the general behaviour of the considerable model. The pressure one of the variables, that is divided to two terms $p = p^C + p^Q$ (classical (p^C) and quantum (p^Q) pressure) has been investigated by Gardner and Haas (1994, 2005). In the momentum equation the classical pressure rises in the form (∇p), while the quantum pressure rises in the form $\boldsymbol{Q} = \frac{\hat{h}^2}{2m_e m_i}\rho\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)$, where $\hat{h}$ is the Plank constant, m_e is the mass of electron and m_i is the mass of ion. One of the important model that rises in hydrodynamic plasma, is called the Rayleigh-Taylor instability problem and has been investigated by Lord Rayleigh (1882) and G. I. Taylor (1950).

Two models are used to study quantum plasma systems. The first one is the Wigner-Poisson and the other is the Schrodinger-Poisson approaches. These have been widely used to describe the statistical and hydrodynamic behaviour of the plasma particles at quantum scale in quantum plasmas. The quantum hydrodynamic model was introduced by Gardner (1994) for semiconductor physics to describe the transport of charge, momentum and energy in plasmas. Several studies were analysed both analytically and numerically in plasma with quantum corrections. For example Haas et al. (2000) studied a quantum multi-stream model for one and two stream plasma instabilities. Bengt Eliasson et al. (2010) and by employing the Wigner-Poisson model, they studied the dispersion properties of electrostatic oscillations in quantum plasmas for different parameters ranging from semiconductor plasmas to typical metallic electron densities and densities corresponding to compressed matter and dense astrophysical objects.

The linear quantum growth rate of a finite layer plasma in which the density is continuously stratified exponentially along the vertical is studied by Goldston and Rutherford (1997). The effect of the quantum term on Rayleigh-Taylor instability of stratified plasma layer through a porous medium is studied by Hoshoudy (2009).

The fluids have been considered to be Newtonian in all the above studies. With the growing importance of the non-Newtonian fluids in modern technology and industries, the investigations of such fluids are desirable. There are many elastico-viscous fluids. But we are interested in Walters' (model B'). Walters' (1960) has proposed a theoretical model for such elastico-viscous fluid. Molten plastics, petroleum oil additives and whipped cream are examples of incompressible viscoelastic fluids. The mixture of polymethyl methacrylate and pyridine at $25^0 C$ containing 30.5 grams of polymer per litre behaves very nearly as the Walters' (model B') viscoelastic fluid and which is proposed by Walters' (1962). Previous work on the effects of incompressible quantum plasma on Rayleigh-Taylor instability of Oldroyd model through a porous medium has been investigated by Hoshoudy (2011), where the author has shown that both maximum $k^{\circ}_{\max}$ and critical k°_{c} points for the instability are unchanged by the addition of the strain retardation and the stress relaxation. All growth rates are reduced in the presence of porosity of the medium, the medium permeability, the strain retardation time and the stress relaxation time. Sunil et al. (2004) have investigated theoretically and analytically the stability of stratified Walters' (model B') viscoelastic fluid in stratified porous medium. Sharma et al. (2014) have studied the numerical investigations of a stability of stratified viscoelastic Walters' (model B′) fluid/plasma in the presence of quantum physics saturating a porous medium. This paper aims at numerical analysis of the effect of the quantum mechanism on Rayleigh-Taylor instability for a finite thickness layer of incompressible viscoelastic plasma. An ideal incompressible magnetized plasma described by the Quantum magnetohydrodynamics (QMHD) model, where in the Rayleigh-Taylor instability by ignoring shear flow or ablation effects, has been studied in quantum magnetized viscous plasma by Hoshoudy (2011a). External magnetic field effects on the Rayleigh-Taylor instability in an inhomogeneous rotating quantum plasma has been studied by Hoshoudy (2012). Later on, Hoshoudy (2013) has investigated quantum effects on Rayleigh-Taylor instability of a Plasma-Vacuum. Recently, Rayleigh-Taylor instability in a magnetized plasma has been investigated by Hoshoudy (2014).

The effect of incompressible quantum plasma on Rayleigh-Taylor instability of Oldroyd model through a porous medium has been investigated by Hoshoudy (2011b), in which it has been shown that both maximum and critical wave numbers for the instability are unchanged due to the strain-retardation and the stress-relaxation. This paper is devoted to examine the stability of a stratified viscoelastic Walters′ (model B′) fluid in hydromagnetics in the presence of quantum physics and is an extension of the research work by Hoshoudy (2010) on the quantum effects on Rayleigh-Taylor instability of incompressible plasma in a vertical magnetic field and it has been found that the presence of vertical magnetic field beside the quantum effect has brought more stability on the growth rate of unstable configuration.

2. Mathematical formulation of the problem and perturbation equations

The initial stationary state whose stability we wish to examine is that of an infinitely electrically conducting incompressible, heterogeneous infinitely extending viscoelastic Walters' (model B$'$) fluid/plasma of thickness d bounded by the rigid planes $z = 0$ and $z = d$, of variable density, kinematic viscosity, kinematic viscoelasticity, magnetic field and quantum pressure arranged in horizontal strata of electrons and immobile ions in a homogeneous, saturated porous medium with the Oberbeck-Boussinesq approximation for density variation, so that the free surfaces are almost horizontal. The fluid is acted on by gravity force $\boldsymbol{g}(0, 0, -g)$ and the plasma is immersed in a uniform vertical magnetic field $\boldsymbol{H}(0, 0, H)$.

The equations of motion, continuity, condition of incompressibility, Gauss divergence equation and magnetic induction equations are [Chandrasekhar (1961), Walters' (1960), Hoshoudy (2009)]

$$\rho\left[\frac{\partial}{\partial t} + (\nabla.\boldsymbol{u})\right]\boldsymbol{u} = -\nabla p + \rho\boldsymbol{g} + \frac{\mu_e}{4\pi}(\nabla \times \boldsymbol{H}) \times \boldsymbol{H} + \left(\mu - \mu'\frac{\partial}{\partial t}\right)\nabla^2\boldsymbol{u} + \boldsymbol{Q}, \tag{1}$$

$$\nabla.\boldsymbol{u} = 0, \tag{2}$$

$$\frac{\partial\rho}{\partial t} + (\boldsymbol{u}.\nabla)\rho = 0, \tag{3}$$

$$\nabla.\boldsymbol{H} = 0 \tag{4}$$

$$\frac{\partial\boldsymbol{H}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{H}), \tag{5}$$

where $\boldsymbol{u},\ \rho,\ p,\ \mu,\ \mu',\ k_1,\ \mu_e, \boldsymbol{H},\ \boldsymbol{Q}$ represent fluid velocity, density, pressure, viscosity, viscoelasticity, medium permeability, magnetic permeability, magnetic field and Bohr vector potential, respectively. Equation (3) ensures that the density of every particle remains unchanged as we follow it with its motion.

The equilibrium profiles are expressed in the form

$$\boldsymbol{u} = (0,0,0),\ \rho = \rho_0(z),\ p = p_0(z),\ \boldsymbol{H} = H_0(z)\ \text{ and }\ \boldsymbol{Q} = \boldsymbol{Q}_0(z). \tag{6}$$

To investigate the stability of hydromagnetic motion, it is necessary to see how the motion responds to a small fluctuation in the value of any flow of the variables. Infinitesimal perturbations are superimposed on the steady state and let $\boldsymbol{u}(u, v, w)$, $\delta\rho$, δp, $\boldsymbol{h}\left(h_x, h_y, h_z\right)$, $\delta\boldsymbol{Q}\left(Q_{x1}, Q_{y1}, Q_{z1}\right)$ denote respectively, infinitesimally small perturbations in fluid velocity $\boldsymbol{u}\ (0, 0, 0)$, density ρ, pressure p, magnetic field $\boldsymbol{H}$ and quantum pressure $\boldsymbol{Q}$.

Using these perturbations and linear theory, equations (1) - (5) in the linearized perturbation form become

$$\rho_0\frac{\partial\boldsymbol{u}}{\partial t} = -\nabla\delta p + \boldsymbol{g}\delta\rho + \frac{\mu_e}{4\pi}[(\nabla \times \boldsymbol{H_0}) \times \boldsymbol{h} + (\nabla \times \boldsymbol{h}) \times \boldsymbol{H_0}] + \left(\mu - \mu'\frac{\partial}{\partial t}\right)\nabla^2\boldsymbol{u} + \delta\boldsymbol{Q}\ , \tag{7}$$

$$\nabla.\boldsymbol{u} = 0, \tag{8}$$

$\frac{\partial}{\partial t}\delta\rho + w\frac{d\rho_0}{dz} = 0,$ (9)

$\nabla.\boldsymbol{h} = 0,$ (10)

$\frac{\partial \boldsymbol{h}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{H_0}),$ (11)

where

$\delta\boldsymbol{Q} =$
$\frac{\hat{h}^2}{2m_e m_i}\Big[\frac{1}{2}\nabla(\nabla^2\delta\rho) - \frac{1}{2\rho_0}\nabla\delta\rho\nabla^2\rho_0 - \frac{1}{2\rho_0}\nabla\rho_0\nabla^2\delta\rho + \frac{\delta\rho}{2\rho_0^2}\nabla\rho_0\nabla^2\rho_0 - \frac{1}{2\rho_0}\nabla(\nabla\rho_0\nabla\delta\rho) +$
$\frac{\delta\rho}{4\rho_0^2}\nabla(\nabla\rho_0)^2 + \frac{1}{2\rho_0^2}(\nabla\rho_0)^2\nabla\delta\rho + \frac{1}{\rho_0^2}(\nabla\rho_0\nabla\delta\rho)\nabla\rho_0 - \frac{\delta\rho}{\rho_0^3}(\nabla\rho_0)^3\Big].$

Since the boundaries are assumed to be rigid. Therefore the boundary conditions appropriate to the problem are

$w = 0, Dw = 0$ at $z = 0$ and $z = d$, on a rigid surface. (12)

All the physical perturbed quantities are ascribed describing the perturbation dependence on x, y and t of the forms

$f(z)\ exp\ i\big(k_x x + k_y y - nt\big),$ (13)

Now, using the expression (13), the equations (7)-(11) reduce to

$-in\rho_0 u = -ik_x\delta p + \frac{\mu_e}{4\pi}\Big[H_0(z)\Big(\frac{\partial h_x}{\partial z} - ik_x h_z\Big) + h_x\frac{\partial H_0(z)}{\partial z}\Big] + \{\mu + in\,\mu'\}(D^2 - k^2)u +$
$\bar{Q}_{x1}$, (14)

$-in\rho_0 v = -ik_y\delta p + \frac{\mu_e}{4\pi}\Big[H_0(z)\Big(\frac{\partial h_y}{\partial z} - ik_y h_z\Big) + h_y\frac{\partial H_0(z)}{\partial z}\Big] + \{\mu + in\,\mu'\}(D^2 - k^2)v +$
$\bar{Q}_{y1}$, (15)

$-in\rho_0 w = -D\delta p - g\delta\rho + \{\mu + in\mu'\}(D^2 - k^2)w + \bar{Q}_{z1},$ (16)

$ik_x u + ik_y v + Dw = 0,$ (17)

$in\ \delta\rho = wD\rho_0,$ (18)

$ik_x h_x + ik_y h_y + Dh_z = 0,$ (19)

$-\ in\ h_x = HDu$, (20)

$-\ in\ h_y = HDv$, (21)

$-\ in\ h_z = HDw$, (22)

where

$\bar{Q}_{x1} = \frac{\hat{h}^2}{2nm_e m_i}\Big[\frac{1}{2}D\rho_0 D^2 w + \Big\{D^2\rho_0 - \frac{1}{2\rho_0}(D^2\rho_0)^2\Big\}Dw + \Big\{\frac{1}{2}D^3\rho_0 - \frac{1}{\rho_0}D\rho_0 D^2\rho_0 -$
$\frac{k^2}{2}D\rho_0 + \frac{1}{2\rho_0^2}(D\rho_0)^3\Big\}w\Big],\ \bar{Q}_{y1} = \frac{k_y}{k_x}\bar{Q}_{x1},$

$$\bar{Q}_{z1} = \frac{\hbar^2}{2nm_e m_i}\Big[\frac{1}{2}D\rho_0 D^3 w + \Big\{\frac{3}{2}D^2\rho_0 - \frac{1}{\rho_0}(D^2\rho_0)^2\Big\}D^2 w + \Big\{\frac{1}{2}D^3\rho_0 - \frac{1}{\rho_0}D\rho_0 D^2\rho_0 - \frac{k^2}{2}D\rho_0 + \frac{3}{2\rho_0^2}(D\rho_0)^3\Big\}Dwk^2 + \frac{1}{2}D^4\rho_0 - \frac{1}{\rho_0}D\rho_0 D^3\rho_0 - \frac{k^2}{2}D^2\rho_0 - \frac{1}{\rho_0}(D^2\rho_0)^2 + \frac{5}{2\rho_0^2}(D\rho_0)^2 D^2\rho_0 + \frac{k^2}{2\rho_0}(D\rho_0)^2 - \frac{1}{\rho_0^2}(D\rho_0)^4\Big]. \quad (22a)$$

Multiplying equation (14) by $-ik_x$ and equation (15) by $-ik_y$, adding and using equations (17), (19) – (22), we obtain

$$\rho n Dw = ik^2\delta p + k_x\bar{Q}_{x1} + k_y\bar{Q}_{y1} - \frac{\mu_e H^2}{4\pi n}(D^2 - k^2)Dw + i(\mu + \mu' i\, n)(D^2 - k^2)Dw. \quad (23)$$

Multiplying equation (14) by $-ik_y$ and equation (15) by ik_x, adding and using equation (17), (19) – (22), we obtain

$$\rho n\xi = -k_x\bar{Q}_{y1} + k_y\bar{Q}_{x1} - \frac{\mu_e H^2}{4\pi n}D^2\xi + i(\mu + \mu' in)(D^2 - k^2)\xi, \quad (24)$$

where $\xi = ik_x v - ik_y u$ is the z-component of vorticity,

Since $k_x\bar{Q}_{y1} = k_y\bar{Q}_{x1}$, therefore, equation (24) implies that $\xi = 0$.

Eliminating $\delta\rho$, δp and $\bar{Q}_{z1}$ from the equations (22a), (16) and (23) the characteristic equation in w obtained is

$$[\rho_0 k^2\{-in - (\nu + \nu' in)(D^2 - k^2)\}]w - [-in - (\nu + \nu' in)(D^2 - k^2)](D\rho_0)Dw + \frac{\mu_e}{4\pi}\Big[4H_0(z)D^2H_0(z) + 4\big(DH_0(z)\big)^2 - k^2\big(H_0^2(z)\big)\Big]D^2 w - \rho_0[-in - (\nu + \nu' in)(D^2 - k^2)]D^2 w + \frac{\mu_e}{4\pi}[H_0(z)D^3H_0(z) + 3DH_0(z)D^2H_0(z) - 2k^2 H_0(z)DH_0(z)]Dw + \frac{\mu_e}{4\pi}\Big[\big(H_0(z)\big)^2 D^4 w + 5H_0(z)DH_0(z)D^3 w\Big] + \frac{gk^2}{in}(D\rho_0)w + \frac{k^2}{in}\Big(\frac{\hbar^2}{4m_e m_i}\Big)\Big[\frac{1}{\rho_0}(D\rho_0)^2 D^2 w - \frac{1}{\rho_0^2}(D\rho_0)\{(D\rho_0)^2 - 2\rho_0 D^2\rho_0\}Dw - \frac{k^2}{\rho_0}(D\rho_0)^2 w\Big] = 0. \quad (25)$$

3. The case of exponentially varying stratifications

Now the case of incompressible continuously stratified viscoelastic plasma layer is considered in a porous medium in which the density, viscosity, viscoelasticity and quantum pressure are assumed to vary exponentially about the vertical and are given by

$$\rho_0(z) = \rho_0(0)exp\left(\frac{z}{L_D}\right),\ \mu_0(z) = \mu_0(0)exp\left(\frac{z}{L_D}\right),\ \mu_0'(z) = \mu_0'(0)exp\left(\frac{z}{L_D}\right),$$
$$H_0(z) = H_0(0)exp\left(\frac{z}{2L_D}\right),\ n_{q_0}(z) = n_{q_0}(0)exp\left(\frac{z}{L_D}\right), \quad (26)$$

where $\rho_0(0)$, $\mu_0(0)$, $\mu_0'(0)$, $H_0(0)$, $n_{q_0}(0)$ and L_D are constants.

Using the stratifications given by expression (26), the characteristic equation (25) yields that

$$V_A^2\left[D^4w+\frac{2}{L_D}D^3w\right]+\left[in\{-in-(\nu+\nu' in)(D^2-k^2)\}-n_q^2+V_A^2\left(\frac{5}{4L_D^2}-k^2\right)\right]D^2w+\frac{1}{L_D}\left[in\{-in-(\nu+\nu' in)(D^2-k^2)\}-n_q^2+V_A^2\left(\frac{1}{2L_D^2}-k^2\right)\right]Dw-k^2\left[in\{-in-(\nu+\nu' in)(D^2-k^2)\}-n_q^2+\frac{g}{L_D}\right]w=0, \tag{27}$$

where $n_q^2=\frac{\hat{h}^2k^2}{4m_em_iL_D^2}$, $V_A^2=\frac{\mu_e\left(H_0^2(z)\right)}{4\pi}$ represent quantum pressure and the square of the Alfvén velocity, respectively.

Differentiating w.r.t. z and using equation (17) the boundary conditions (12) yields that

$$D^2w=0 \text{ at } z=0 \text{ and } z=d. \tag{28}$$

The exact solutions of the eigen-value problem (25) satisfying the boundary conditions (28) are chosen to be

$w=sin\left(\frac{n'\pi}{d}z\right)exp(\lambda z)$, where n' and λ are positive integers.

Using this solution the equation (24), yields that

$$V_A^2\Bigg[\left(\frac{n'\pi}{d}\right)^4 sin\left(\frac{n'\pi}{d}z\right)-4\lambda\left(\frac{n'\pi}{d}\right)^3 cos\left(\frac{n'\pi}{d}z\right)-6\lambda^2\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+3\lambda^3\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda^4 sin\left(\frac{n'\pi}{d}z\right)+\frac{2}{L_D}\left\{-\left(\frac{n'\pi}{d}\right)^3 cos\left(\frac{n'\pi}{d}z\right)-2\lambda\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+2\lambda^2\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+\lambda^2\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda^3 sin\left(\frac{n'\pi}{d}z\right)\right\}\Bigg]+\left[in(-in)-n_q^2+V_A^2\left(\frac{5}{4L_D^2}-k^2\right)\right]\left[-\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+2\lambda\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)\right]-in(\nu+\nu' in)\left[\left(\frac{n'\pi}{d}\right)^4 sin\left(\frac{n'\pi}{d}z\right)-4\lambda\left(\frac{n'\pi}{d}\right)^3 cos\left(\frac{n'\pi}{d}z\right)-6\lambda^2\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+3\lambda^3\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda^4 sin\left(\frac{n'\pi}{d}z\right)\right]+in(\nu+\nu' in)k^2\left[-\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+2\lambda\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)\right]+\frac{1}{L_D}\left[in(-in)-n_q^2+V_A^2\left(\frac{1}{2L_D^2}-k^2\right)\right]\left[\left(\frac{n'\pi}{h}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda sin\left(\frac{n'\pi}{d}z\right)\right]-in(\nu+\nu' in)\left\{-\left(\frac{n'\pi}{d}\right)^3 cos\left(\frac{n'\pi}{d}z\right)-2\lambda\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+2\lambda^2\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right)+\lambda^2\left(\frac{n'\pi}{d}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda^3 sin\left(\frac{n'\pi}{d}z\right)\right\}+in(\nu+\nu' in)k^2\left[\left(\frac{n'\pi}{h}\right)cos\left(\frac{n'\pi}{d}z\right)+\lambda sin\left(\frac{n'\pi}{d}z\right)\right]-k^2\left[in(-in)+ina^2(\nu+\nu' in)-\right.$$

$$n_q^2 + \frac{g}{L_D}\Big] sin\left(\frac{n'\pi}{d}z\right) + ink^2(\nu + \nu' in)\left[-\left(\frac{n'\pi}{d}\right)^2 sin\left(\frac{n'\pi}{d}z\right) + 2\lambda\left(\frac{n'\pi}{d}\right) cos\left(\frac{n'\pi}{d}z\right)\right] = 0 \qquad (29)$$

Equating the coefficients of $sin\left(\frac{n'\pi}{d}z\right)$ and $cos\left(\frac{n'\pi}{d}z\right)$ from equation (29), one obtains

$$V_A^2\left[\left(\frac{n'\pi}{d}\right)^4 - 6\lambda^2\left(\frac{n'\pi}{d}\right)^2 + \lambda^4 + \frac{2}{L_D}\left\{-2\lambda\left(\frac{n'\pi}{d}\right)^2 + \lambda\left(\frac{n'\pi}{d}\right)^2 + \lambda^3\right\}\right] - \left(\frac{n'\pi}{d}\right)^2\Big[n^2 - n_q^2 + V_A^2\left(\frac{5}{4L_D^2} - k^2\right)\Big] + \frac{\lambda}{L_D}\left[n^2 - n_q^2 + V_A^2\left(\frac{1}{2L_D^2} - k^2\right)\right] - k^2\Big[n^2 + ink^2(\nu + \nu' in) - n_q^2 + \frac{g}{L_D}\Big] - in(\nu + \nu' in)\left[\left(\frac{n'\pi}{d}\right)^4 - 6\lambda^2\left(\frac{n'\pi}{d}\right)^2 + \lambda^4\right] - in(\nu + \nu' in)k^2\left(\frac{n'\pi}{d}\right)^2 + \lambda\, i\, n(\nu + \nu' in)k^2 - in(\nu + \nu' in)\left\{-2\lambda\left(\frac{n'\pi}{d}\right)^2 + \lambda\left(\frac{n'\pi}{d}\right)^2 + \lambda^3\right\} - ink^2(\nu + \nu' in)\left(\frac{n'\pi}{d}\right)^2 = 0, \text{ and} \qquad (30)$$

$$V_A^2\left[-4\lambda\left(\frac{n'\pi}{d}\right)^3 + 3\lambda^3\left(\frac{n'\pi}{d}\right) + \frac{2}{L_D}\left\{-\left(\frac{n'\pi}{d}\right)^3 + 2\lambda^2\left(\frac{n'\pi}{d}\right) + \lambda^2\left(\frac{n'\pi}{d}\right)\right\}\right] + 2\lambda\left(\frac{n'\pi}{d}\right)\Big[n^2 - n_q^2 + V_A^2\left(\frac{5}{4L_D^2} - k^2\right)\Big] + \frac{1}{L_D}\left[n^2 - n_q^2 + V_A^2\left(\frac{1}{2L_D^2} - k^2\right)\right]\left(\frac{n'\pi}{d}\right) - in(\nu + \nu' in)\left[-4\lambda\left(\frac{n'\pi}{d}\right)^3 + 3\lambda^3\left(\frac{n'\pi}{d}\right)\right] + 2\lambda\left(\frac{n'\pi}{d}\right) in(\nu + \nu' in)k^2 - in(\nu + \nu' in)\left\{-\left(\frac{n'\pi}{d}\right)^3 + 2\lambda^2\left(\frac{n'\pi}{d}\right) + \lambda^2\left(\frac{n'\pi}{d}\right)\right\} + in(\nu + \nu' in)k^2\left(\frac{n'\pi}{d}\right) = 0. \qquad (31)$$

Now introducing the non-dimensional quantities

$$n^{*2} = \frac{n^2}{n_{pe}^2},\ n_q^{*2} = \frac{n_q^2}{k^2 L_D^2 n_{pe}^2},\ \nu^* = \frac{\nu}{n_{pe}}, \nu'^* = \frac{\nu'}{n_{pe}}, k_1^* = \frac{k_1}{n_{pe}}, d^{*2} = \frac{d^2}{L_D^2}, k^{*2} = k^2 L_D^2,\ V_A^{*2} = \frac{V_A^2}{n_{pe}^2 L_D}, \lambda^{*2} = \lambda^2 L_D^2,\ n_{pe} = \left(\frac{\rho_0 e^2}{m_e^2 \varepsilon_0}\right)^{1/2},\ g^* = \frac{g}{n_{pe}^2 L_D}.$$

where $n_{pe} = \left(\frac{\rho e^2}{m_e^2 \varepsilon_0}\right)^{1/2}$, is the plasma frequency.

The equation (30) after dropping the asterisk for our convenience and in the absence of vertical magnetic field, $V_A^2 = 0$; equation (31) yields that $\lambda = \frac{1}{2}$ and substituting this value of λ in equation (30), the dispersion relation so obtained is

$$A_1(i\,n)^2 + A_2(i\,n) - A_3 = 0, \qquad (32)$$

where, the constants $A_1 - A_3$ containing large number of terms so we omit here.

Since $n = n_r + in_i$ and in the case of $n_r = 0$ and $n_i \neq 0$ (stable oscillations), then the equation (32) becomes

$$A_1 {n_i}^2 - A_2 n_i - A_3 = 0, \qquad (33)$$

which is the required dispersion relation studying the effects of magnetic field, viscosity, viscoelasticity and quantum pressure, respectively.

4. Numerical results and discussion

Numerical computations are carried out using the dispersion relation described by equation (33), using the software Mathematica version 5.2 to look into the effect of various factors on the instability of the considered system. This is to find the role of the quantum pressure and magnetic field on the square of the normalized growth rate of the unstable mode of perturbation for fixed permissible values of the dimensionless parameters $\nu' = 0.2, n_q = 0.6, V_A^2 = 0.2, n' = 1, d = 1, g = 9.8$ and $\nu = 0.1$. Pertaining results are presented in figures 2 and 3.

Figure 2 shows the variations of the square of the normalized growth rate n_i^2 with respect to the square normalized wavenumber k^2 for three different values of square of the Alfvén velocity $V_A^2 = 0.2, 0.3, 0.5$. It is evident from the graph that the growth rates increases for $k \leq 2.9\ (V_A^2 = 0.2),\ k \leq 5\ (V_A^2 = 0.3),\ k \leq 7\ (V_A^2 = 0.5)$ showing thereby the destabilizing effect whereas the growth rates decrease for $7 \leq k \leq 30$ implying thereby the stabilizing effect of Alfvén velocity on the system.

Figure 3 corresponds to the three different values of quantum plasma $n_q = 0.2, 0.3, 0.5$, respectively. It is clear from the graphs that the growth rates increase for $k \leq 6$, showing thereby the destabilizing effect, whereas the growth rates decrease for $k > 6$, implying thereby the stabilizing effect of quantum pressure on the system.

It is clear from the figures 2 - 3 that the simultaneous presence of magnetic field implying thereby the large enough stabilizing effect of the quantum pressure on the system.

5. Conclusions

The stability of stratified viscoelastic Walters' (model B$'$) fluid/plasma in hydromagnetics in the presence of quantum physics has been studied. The principal results of the analysis are as follows:

i) The magnetic field has a stabilizing effect on the system under certain wavenumber band.

ii) The effect of quantum pressure with the simultaneous presence of magnetic field is more stabilizing.

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7. Figures

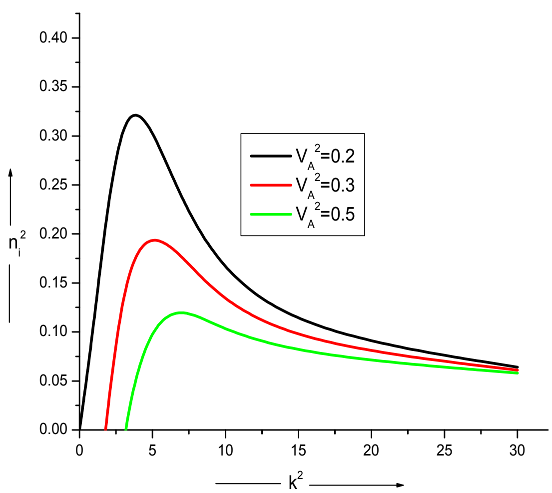

Figure 2: Variations of the square of normalized growth rate n_i^2 versus the square normalized wavenumber k^2 for three different values of the square of the Alfvén velocity $V_A^2 = 0.2,\ 0.3,\ 0.5$.

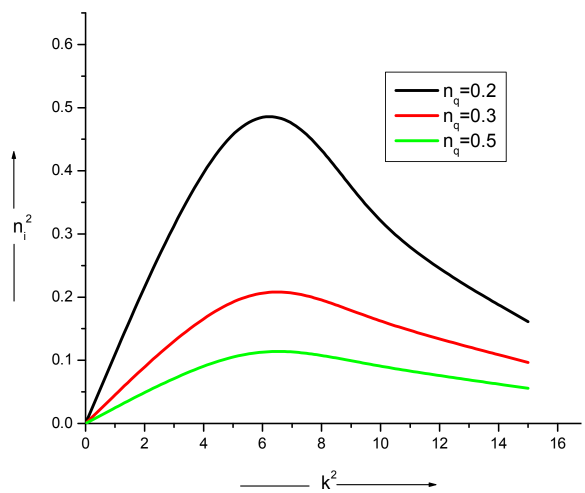

Figure 3: Variations of the square of normalized growth rate n_i^2 versus the square normalized wavenumber k^2 for three different values of quantum pressure $n_q = 0.2,\ 0.3,\ 0.5$.

Soret Driven Double-Diffusive Steady Magnetoconvection with Rigid and Impervious Boundaries

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ABSTRACT

In the present paper, the problem of double-diffusive steady convection in the presence of Soret effect and magnetic field has been studied. The eigen value equations governing the problem for stationary convection for rigid and impervious boundaries have been casted into a mathematical tractable form by using certain linear transformations. An expression for Rayleigh number for stationary convection using variational principle is obtained and, consequently, a necessary condition for the validity of principle of exchange of stabilities is obtained by using trial function satisfying the essential boundary conditions from the minimum property of the functional. The effects of Soret parameter, Chandrasekhar number and Lewis number on stationary convection have been discussed and it is found that, the Soret parameter has both stabilizing as well as destabilizing effect (depending upon the sign) on the stationary double diffusive convection, whereas the Chandrasekhar number and the Lewis number have stabilizing effect on the stability of the system.

Keywords: Double diffusive convection, Soret effect, Stationary convection, Magnetic field. Impervious boundaries.

1. INTRODUCTION

The problem of thermal instability in a thin layer of a Newtonian fluid with single diffusive (heat) component in the force field of gravity has been extensively studied by many authors, under the varying assumptions of hydrodynamics and hydromagnetics. The main objective of the studies related to thermal instability, in particular, is to determine the value of the Rayleigh number which characterizes the stability or instability of the system or to derive certain criteria for the onset of instability through convection. The detailed account of such analysis is given in the treatise by Chandrasekhar [5].

The study of convective motions in the presence of two diffusing components with different diffusivities is an interesting phenomena in the field of convection, known as double-diffusive convection or thermohaline convection. Double-diffusive convection involves motions driven by two different density gradients diffusing at different rates (Mojtabi and Charrier-Mojtabi [11]). The interesting effects of double-diffusive convection are due to the sharp contrasts between thermal and salt diffusivities and were first observed by Stern [15] in 1960 and by Veronis [18] in 1965. In double diffusive system, the convection starts due to variations in temperature and solute concentrations both. The flux of heat caused by concentration gradient is termed as Dufour effect, whereas, the flux of mass caused by temperature gradient is known as the Soret effect.

Due to the cross diffusion effect (Soret and Dufour effects), each property gradient has a significant influence on the flux of the other property. According to Schechter et. al. [14], Bergeron et. al. [4] and Straughan and Hutter [16], the Dufour coefficient is of order of magnitude smaller than the Soret coefficient in liquids, and the corresponding contribution to the heat flux may be neglected in liquids in comparison to the Soret effect. The study of Soret driven double diffusive convection has received much attention over the years due to its numerous fundamental and industrial applications in various fields such as high quality crystal production oceanography, solidification of molten alloys, astrophysics and engineering. For a broader view on the subject, one may refer to Hurle and Jakeman [8], Malashetty & Gaikwad [10] and Dhiman and Goyal [7].

Chandrasekhar [6] and Banerjee et. al. [3] have investigated the effect of magnetic field on the stability of Bénard convection problem in detail. Banerjee et. al. [2] studied a more general problem, namely, magnetohydrodynamic thermohaline convection problem and derived a characterization theorem. N. Rudraiah [12, 13] has made a study of double diffusive magnetoconvection and shown that magnetic field destabilizes the double diffusive system under certain conditions. Takashima [17] studies the effect of magnetic field on convective instability in a horizontal layer of two component fluid with Soret effect and it has been established that even if a magnetic field is present, the presence of solute plays a prominent role through the Soret effect and that even if the solute is present, the magnetic field inhibits the onset of instability.

Most of the authors have dealt the convection problems in a horizontal layer for the *unrealistic* case of both *dynamically free* boundaries in which no tangential stress acts. For the realistic case of both rigid bounding surfaces, the exact solutions in closed forms are not obtainable because of the mathematical complexities in the governing eigen value equations. Further, for the solutions when the binary fluids are subjected to Soret effect, the boundary conditions on concentration, in view of the solid boundaries, must be impervious (Bahloul et. al. [1]). Dhiman and Goyal recently studied the stability of Soret driven double-diffusive convection problem analytically for the case of rigid, impervious and thermally perfectly conducting boundary conditions using Variational principle.

Motivated by the above discussions and the role of rigid and impervious bounding surfaces on the onset of convection, in the present analysis the effect of magnetic field on the onset of Soret driven double-diffusive steady convection with rigid and impervious boundaries has been studied. The eigen value equations governing the problem under consideration has been transformed into a mathematical tractable form for the variational treatment using some indigenous linear transformations. The variational principle has been established for the problem and using the minimum property of the functional, an expression for Rayleigh number has been obtained. The effects of Soret parameter, Chandrasekhar number and Lewis number on stationary convection have been discussed.

2. PHYSICAL CONFIGURATION AND GOVERNING EQUATIONS

Consider a electrically conducting viscous, quasi-incompressible two component fluid of infinite horizontal extension and finite vertical depth statically confined between two horizontal boundaries $z = 0$ and $\mathrm{z = d}$ which are respectively maintained at uniform temperatures T_0 and $T_1 \left(T_1 < T_0\right)$ and uniform concentrations C_0 and $C_1 \left(C_1 < C_0\right)$ in the presence of uniform magnetic field acting in the vertical direction in the force field of gravity. Both the boundaries are assumed to be a rigid, impervious and perfectly heat conducting.

Following the usual steps of the linear stability analysis [5, 7], we obtain the following system of non-dimensional linearized perturbation equations;

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w = Ra^2\theta - R'a^2\phi - QD\left(D^2 - a^2\right)h_z \tag{1}$$

$$\left(D^2 - a^2 - p\right)\theta = -w \tag{2}$$

$$\left(D^2 - a^2 - \frac{p}{\tau}\right)\phi = -\frac{w}{\tau} + \left(D^2 - a^2\right)\theta \tag{3}$$

$$\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw \tag{4}$$

We consider the case where both the boundaries are rigid, impervious and perfectly conducting. Thus, the appropriate boundary conditions for the present problem are;

$$w = Dw = \theta = h_z = 0,\ D\phi - D\theta = 0 \ ; \text{at z = 0 and z = 1.} \tag{5}$$

In the forgoing equations; z is the real independent variable, $D \equiv d/dz$ is the differentiation with respect to z, a^2 is the square of the wave number, σ is the Prandtl number, σ_1 is the coefficient of electrical conductivity, τ is the Lewis number, R is the thermal Rayleigh number, R' is the solutal Rayleigh number, Q is the Chandrasekhar number, $p(= p_r + ip_i)$ is the complex growth rate and w, θ, ϕ and h_z are the perturbations in the vertical velocity, temperature, concentration and magnetic field respectively. The system of equations (1)-(4) together with the boundary conditions (5) constitutes an eigen value problem for R for given values of other parameters, namely R', σ, τ and a^2. Further, a given state of system is stable, neutral or unstable according as p_r is negative, zero or positive. Further, if $p_r = 0$ implies $p_i = 0$ for all wave numbers a^2, then the principle of exchange of stability (PES) is valid, otherwise, we have overstability at least when the instability sets in as a certain modes.

3. MATHEMATICAL ANALYSIS

When the instability sets in as stationary convection i.e. when PES is valid, we have p= 0, therefore, the equations (1)–(4) and boundary conditions (5) becomes

$$\left(D^2 - a^2\right)^2 w = Ra^2\theta - R'a^2\phi - QD\left(D^2 - a^2\right)h_z \tag{6}$$

$$\left(D^2 - a^2\right)\theta = -w \tag{7}$$

$$\left(D^2 - a^2\right)\phi = -\frac{w}{\tau} + \left(D^2 - a^2\right)\theta \tag{8}$$

$$\left(D^2 - a^2\right)h_z = -Dw \tag{9}$$

together with the boundary conditions

$$\left.\begin{aligned} w = Dw = \theta = 0 \\ D\phi - D\theta = 0 \end{aligned}\right\} \quad \text{at } z = 0 \text{ and } z = 1. \tag{10}$$

and

Now, redefining θ and ϕ as follows

$$F = \frac{g\alpha\beta d^4 a^2}{\nu\kappa}\theta, \; G = \frac{g\alpha'\beta' d^4}{\nu\kappa}a^2\phi \tag{11}$$

and using the linear relation

$$M = \gamma F + G, \tag{12}$$

equations (6)–(9) and boundary conditions (10) assume the following form;

$$\left(D^2-a^2\right)^2 w=(1+\gamma)F-M-QD\left(D^2-a^2\right)h_z \tag{13}$$

$$\left(D^2-a^2\right)F=-Ra^2 w \tag{14}$$

$$\left(D^2-a^2\right)M=-\frac{R'a^2 w}{\tau} \tag{15}$$

$$\left(D^2-a^2\right)h_z=-Dw \tag{16}$$

$$w=Dw=DM=0 \text{ at } z=0 \text{ and } z=1. \tag{17}$$

In the above equations, the thermal and solutal Rayleigh numbers are related by the expression $\gamma R=-R'$, where γ is called stability ratio (or Soret parameter) and defined as $\gamma=S_T\ N_0(1-N_0)\ \alpha'/\alpha$. The strength of the *Soret* forcing in mixtures is parameterized by the stability ratio, depending on the mixture the *Soret* coefficient can be positive or negative, meaning thereby that solute can be driven toward the hotter, or the colder region. Hence, γ can take both positive and negative values (La-Porta and Surko [9]).

To find the necessary condition for the validity of the PES and consequently the critical Rayleigh number for the present problem, we proceed as follows.

Multiplying equation (14) by F and integrating over z, we get

$$\int_0^1 F\left(D^2-a^2\right)F\,dz=-Ra^2\int_0^1 wF\,dz \tag{18}$$

Inserting the value of F from equation (13) in the right hand side of equation (18), making use of equation (16) in the resulting equation, we obtain

$$\int_0^1 F\left(D^2-a^2\right)F\ dz=\frac{-Ra^2}{1+\gamma}\int_0^1 w\left\{\left(D^2-a^2\right)^2-QD^2\right\}w dz-\frac{Ra^2}{1+\gamma}\int_0^1 wM\ dz \tag{19}$$

which upon using equation (15) in the second integral on right hand side yields

$$\int_0^1 F\left(D^2-a^2\right)F\ dz=\frac{-Ra^2}{1+\gamma}\int_0^1 w\left\{\left(D^2-a^2\right)^2-QD^2\right\}w dz-\frac{\tau R}{(1+\gamma)R'}\int_0^1 M\left(D^2-a^2\right)M\ \ dz$$

Integrating by parts the above equation a suitable number of times, using the boundary conditions (17), we have the following expression

$$R = \frac{(1+\gamma)\int_0^1 \left\{ (DF)^2 + a^2 F^2 \right\} dz}{a^2 \left[\int_0^1 \left\{ (D^2 w)^2 + a^4 w^2 + (2a^2 + Q)(Dw)^2 \right\} dz + \frac{\tau}{R' a^2} \int_0^1 \left\{ (DM)^2 + a^2 M^2 \right\} dz \right]}$$

$$= \frac{(1+\gamma) I_1}{a^2 I_2}, (say) \qquad (20)$$

The above expression for R (the Rayleigh number), which is the ratio of two positive definite integrals, is the required functional for the variational treatment of the problem.

Following the variational method of Chandrasekhar for thermal convection problem and proceeding analogously, we can easily prove the stationary property of the functional R given by expression (20) for the boundary conditions (17) when the quantities on right hand side are evaluated in terms of true characteristic functions. Also the quantity on the right hand side of equation (20) attains its true minimum when F belongs to R_c i.e. the lowest characteristic value of R, namely R_c, is indeed a true minimum. i.e.

$$R_c \leq R = \frac{(1+\gamma) I_1}{a^2 I_2}. \qquad (21)$$

Thus, the above result generalizes the variational method for the problems of Soret driven double-diffusive magnetoconvection with rigid, impervious and heat conducting boundaries.

Now, we shall evaluate the integrals I_1 and I_2 by using the trial functions satisfying the given boundary conditions. Let us consider a trail function

$$w(z) = z^2 (1-z)^2 = z^4 - 2z^3 + z^2 \qquad (22)$$

which obviously satisfies the boundary conditions

$w = 0$ at $z = 0$ and $z = 1$

Now, using equation (22), we have

$$\int \left\{ (D^2 w)^2 + (2a^2 + Q)(Dw)^2 + a^4 w^2 \right\} dz = 0.8 + 0.038a^2 + 0.0015873a^4 + 0.019Q \qquad (23)$$

Inserting $w(z) = z^4 - 2z^3 + z^2$ in equation (14) and solving the resulting equation for $F(z)$, using the relevant boundary conditions given by equation (17), we obtain

$$F(z) = Ra^2\left[-\left(\frac{2}{a^2}+\frac{24}{a^6}\right)\frac{\cosh a\left(\tfrac{1}{2}-z\right)}{\cosh \tfrac{a}{2}}+\frac{z^4+z^2-2z^3}{a^2}+\frac{12z^2+2-12z}{a^4}+\frac{24}{a^6}\right] \quad (24)$$

Now, multiplying both sides of equation (14) by F and integrating the resulting equation over the range of z, we obtain

$$\int_0^1 \left\{F\left(D^2-a^2\right)F\right\}dz = -Ra^2\int_0^1 w(z)F(z).dz\,. \quad (25)$$

Inserting the values of $w(z)$ and $F(z)$ from equations (22) and (24) respectively in the right hand side integral of above equation and integrating the resulting equation a suitable number of times, using the relevant boundary conditions (17), we obtain

$$\int_0^1 \left\{(DF)^2+a^2F^2\right\}dz = R^2a^2\left[-\frac{8}{a^5}\left(1+\frac{12}{a^2}\right).\left\{\left(1+\frac{12}{a^2}\right)\tanh(a/2)-\frac{6}{a}\right\}+\right.$$
$$\left.0.00158-\frac{0.019}{a^2}+\frac{0.8}{a^4}\right] \quad (26)$$

Again, inserting $w(z)=z^4-2z^3+z^2$ in equation (15) and solving the resulting equation for $M(z)$, using the relevant boundary conditions given by equation (17), we obtain

$$M(z) = \frac{R'}{\tau}.\left[\frac{-12\cosh a\left(z-\tfrac{1}{2}\right)}{a^3\sinh(a/2)}+z^4-2z^3+z^2+\frac{12z^2-12z+2}{a^2}+\frac{24}{a^4}\right]. \quad (27)$$

Now, multiplying both sides of equation (15) by M and integrating the resulting equation over the range of z, we obtain

$$\int_0^1 \left\{M\left(D^2-a^2\right)M\right\}dz = -\frac{R'a^2}{\tau}\int_0^1 w(z)M(z).dz\,. \quad (28)$$

Inserting the values of $w(z)$ and $F(z)$ from equations (22) and (27) respectively in the right hand side integral of above equation and integrating the resulting equation a suitable number of times, using the relevant boundary conditions (17), we obtain

$$\int_0^1 \left\{(DM)^2+a^2M^2\right\}dz = \frac{R'^2a^2}{\tau^2}\left[-\frac{12}{a^3}\left\{\left(\frac{4}{a^3}+\frac{48}{a^5}\right)-\frac{24}{a^4}\coth(a/2)\right\}+\right.$$
$$\left.0.00158-\frac{0.019}{a^2}+\frac{0.8}{a^4}\right] \quad (29)$$

Now, substituting the values of various definite integrals from (23), (26) and (29) in equation (20) and upon using the fact that $R' = -\gamma R$, we obtain

$$R = \frac{(1+\gamma)R'^2 K_1}{\gamma^2 K_2} \tag{30}$$

where

$$K_1 = -\frac{8}{a^5}\left(1+\frac{12}{a^2}\right)\left\{\left(1+\frac{12}{a^2}\right)\tan h(a/2) - \frac{6}{a}\right\} + 0.0015873 - \frac{0.019}{a^2} + \frac{0.8}{a^4} \tag{31}$$

and

$$K_2 = \frac{R'}{\tau^2}\left[-\frac{48}{a^6}\left\{\left(1+\frac{12}{a^2}\right) - \frac{6}{a}\coth(a/2)\right\} + 0.0015873 - \frac{0.019}{a^2} + \frac{0.8}{a^4}\right] + 0.8 + 0.038a^2 + 0.0015873a^4 + 0.019Q \tag{32}$$

Now, we assume that Q and R' are of same order of magnitude and therefore, we take $a = O(Q^{\frac{1}{6}})$ and $a = O(R'^{\frac{1}{6}})$. Substituting the values of K_1 and K_2 from equations (31) and (32) in equation (21) and utilizing sufficiently large values of Q in the resulting equation, we obtain

$$R = \frac{Q}{\gamma}\left(1+\frac{1}{\gamma}\right)\left[\frac{0.015873}{0.019 + \dfrac{0.0015873}{\tau}}\right] = \frac{Q}{\gamma}\left(1+\frac{1}{\gamma}\right)\left(\frac{1}{11.97+\tau^{-1}}\right) \tag{33}$$

Using above value of R, inequality (21) yields

$$R_c \le \frac{Q}{\gamma}\left(1+\frac{1}{\gamma}\right)\left(\frac{1}{11.97+\tau^{-1}}\right)$$

which further implies that

$$R_c \le \frac{Q}{\gamma}\left(1+\frac{1}{\gamma}\right) \qquad \text{for large values of Q} \tag{34}$$

which is a necessary condition (dependent upon Soret parameter) for the validity of PES for the onset of stationary convection in Soret-driven double-diffusive convection in the presence of magnetic field when both the bounding surfaces are rigid, impervious and thermally conducting.

Now to study the effect of Soret parameter, Chandrasekhar number and Lewis number on the double diffusive system, we examine the behaviour of $\partial R/\partial\gamma$, $\partial R/\partial Q$ and $\partial R/\partial\tau$ analytically.

From equation (33), we have

$$\frac{\partial R}{\partial\gamma} = -\frac{Q\,(\gamma+2)}{\gamma^3}\left(\frac{1}{11.97+\tau^{-1}}\right)$$

which is positive if $\gamma < 0$ and negative if $\gamma > 0$. Hence, for fixed positive values of Chandrasekhar number and Lewis number, the value of stationary Rayleigh number increases with increasing values of Soret parameter if $\gamma < 0$ and decreases with increasing values of Soret parameter if $\gamma > 0$. Thus, for Soret-driven double-diffusive convection in the presence of magnetic field, the Soret parameter has both stabilizing as well as destabilizing effect on the onset of the stationary convection according as $\gamma < 0$ and $\gamma > 0$.

Further, we can have from equation (33) that

$$\frac{\partial R}{\partial Q} = \frac{\gamma+1}{\gamma^2}\left(\frac{1}{11.97+\tau^{-1}}\right) > 0$$

which implies that for fixed values of Soret parameter and Lewis number, the value of stationary Rayleigh number increases with increasing values of Chandrasekhar number. Thus, for the stationary convection Chandrasekhar number has a stabilizing effect on the double diffusive system.

Also, we can have from equation (33) that

$$\frac{\partial R}{\partial\tau} = \frac{Q(\gamma+1)}{\gamma^2}\left(\frac{1}{11.97\tau+1}\right)^2 > 0$$

which implies that for fixed positive values of Soret parameter and Chandrasekhar number, the value of stationary Rayleigh number increases with increasing values of Lewis number. Thus for the stationary convection Lewis number has a stabilizing effect on the double diffusive system.

4. CONCLUSIONS

In the present analysis, the eigen value problem governing the Soret-driven double-diffusive stationary magnetoconvection problem has been transformed into an eigen value problem which behaves nicely for the variational treatment of the problem. The variational principle for Soret-driven double-diffusive stationary convection problem in the presence of magnetic field with realistic case of rigid*, impervious* and *thermally* *conducting* boundary conditions has been established. A necessary condition for the validity of PES for this general problem utilizing the minimum property of variational principle have been established. Further, a expression for Rayleigh number is obtained as a function of the governing parameters, which characterize the stability of the system. The analysis reveals that the onset of magnetoconvection in double diffusive flow strongly depends upon of the Soret parameter. The effect of various parameters such as Soret parameter, Chandrasekhar number and Lewis number on the onset of stationary convection has been discussed. The following conclusions are drawn from our investigations;

(i) For the case of stationary double diffusive convection in the presence of magnetic field the Soret parameter has both stabilizing as well as destabilizing effect on the double diffusive system according as $\gamma<0$ or $\gamma>0$.

(ii) The Chandrasekhar number and Lewis number has stabilizing effect on the onset of stationary magnetoconvection in the double diffusive system with Soret effect.

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Upper Bounds for The Complex Growth Rate in Magnetohydrodynamic Triply Diffusive Convection with viscosity variations

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Abstract: Upper bounds for the complex growth rate of an arbitrary oscillatory perturbation which may be neutral or unstable in triply diffusive fluid layer heated from below, which is kept under the effect of uniform vertical magnetic field with the viscosity variation effects included are obtained. These results are uniformly valid for quite general nature of the bounding surfaces.

Keywords: Triply diffusive convection, variable viscosity, concentration Rayleigh number, oscillatory motion, Chandrasekhar number.

Introduction

When two stratifying agencies with gravitationally opposite contributions(for instance heat and salt) are present in a viscous fluid, a variety of convective phenomena are found to occur which are known as thermosolutal convection or thermohaline convection or more generally double diffusive convection. Double diffusive convection is now well known and for a broad view of the subject one may referred to Turner [25] and Brandt and Fernando [2].

There are many important hydrodynamical systems in which the density depends on more than two stratifying agencies having different molecular diffusivities. Some examples of these are the earth's core, sea water, solidifying alloys, geothermally heated lakes, magmas and their laboratory models (Turner [24]). Earlier theoretical and experimental studies of the hydro dynamical configurations where the density depends on three stratifying agencies include the work of Griffiths [3-4], Pearlstein et al. [11], Moroz [9], Lopez et al. [8]. In the latter studies, Terrones [22] studied the effects of cross diffusion on the onset of convective instability in a horizontal triply diffusive fluid layer. Straughan and Walker [20] investigated the penetrative convection in a triply diffusive fluid layer. Straughan and Tracey [19] analysed multicomponent convection diffusion with internal heating or cooling in a fluid layer. The long-time behaviour of a triply convective-diffusive fluid mixture saturating a porous horizontal layer has been studied by Rionero [13]. Shivakumara and Kumar [17] investigated the effect of couple stresses on linear and weakly nonlinear stability of a triply diffusive fluid layer. Ryzhkov and Shevtsova [16] analysed the long wave instability of a multicomponent fluid layer with the soret effect included. Rionero [14] studied a triple convective diffusive fluid mixture saturating a porous horizontal layer, heated from below and salted from above and derive sufficient conditions for inhibiting the onset of convection and guaranteeing the global nonlinear stability of the thermal conduction solution. Rionero [15] also studied the multicomponent diffusive convection in porous layer for the more general case when heated from below and salted by m salts partly from above and partly from below. Recently Prakash et al. [12] derived upper bounds for the complex growth rate in triply diffusive convection.

It is, however, to note that, in most of studies cited in the preceding paragraphs, the fluid viscosity is considered as constant with respect to temperature variations. For many fluids, such as glycerine, silicone fluid, petroleum and some molten salts, the variation of viscosity with temperature is often much rapid than that of the other properties. Thus the effect of the variation of the viscosity due to temperature on the stability analysis of any hydrodynamic system must be included to have realistic approach.

The dependence of viscosity on temperature and/or depth for most fluids has prompted several investigations on the onset of convection in hydrodynamics. The effect of large variations of viscosity on thermal convection in a layer of fluid heated from below has been numerically investigated by Torrance and Turcotte [23]. Korenaga and Jordan [7] studied the influence of temperature-and depth-dependent viscosity on the onset of convection in an incompressible fluid cooled from above on the basis of 2-D numerical simulation. Kaddiri et al. [6] studied the effects of temperature-dependent viscosity on the Rayleigh-Benard convection of non-Newtonian power-law fluids confined in a square cavity, heated from bottom and cooled on the top with uniform heat fluxes. Payne and Straughan [10] studied the nonlinear stability of thermal convection in a porous layer when viscosity depends on temperature. Global stability for thermal convection in a couple-stress fluid with temperature and pressure dependent viscosity has been investigated by Sunil and Chaudhary [21]. Banerjee et al. [1] mathematically analyse the stability of generalized Benard problem with a viscosity which is a linear function of depth (on account of thermal effects). Gupta and Kaushal [5] analytically investigated the rotatory hydromagnetic double diffusive convection problems by considering the effects of viscosity variations due to temperature and concentration.

The present communication is primarily motivated by the investigations of Gupta and Kaushal [5] and their work has been extended to magnetohydrodynamic triply diffusive convection problems in the domains of astrophysics and terrestrial physics, wherein the liquid concerned has the property of electrical conduction and the magnetic field is prevalent. The choice of a temperature and concentration dependent viscosity on the pattern of density in double-diffusive and triply-diffusive convection problems has a limitation that viscosity is a linear function of the vertical coordinate [5] which may not necessarily be so in a real physical situation. Thus in the governing equations of the magnetohydrodynamic triply diffusive convection problem viscosity has been taken as an arbitrary function of the vertical coordinate which is in accordance with role of viscosity in Rayleigh-Taylor instability problem. Since the resulting governing non dimensional differential equations have variable coefficient of viscosity contrary to the case wherein viscosity is considered as constant, thus these more general problems introduce extra mathematical complexities. Thus an attempt is made to mathematically tackle problems with more complexities and extending the domain of validity of the earlier results concerning the region of the complex growth rate in the literature of the triply diffusive convection problem with constant viscosity, which are important, especially when at least one boundary is rigid so that exact solutions in the closed form are not obtainable.

Mathematical Formulation and analysis

An infinite horizontal layer filled with a Boussinesq viscous fluid is statically confined between two horizontal boundaries $z = 0$ and $z = d$ (kept under the influence of a uniform vertical magnetic field), maintained at constant temperatures T_0 and $T_1(< T_0)$ and uniform solute concentrations S_{10}, S_{20} and $S_{11}(< S_{10})$, $S_{21}(< S_{20})$ at the lower and upper boundaries respectively. Let the origin be taken on the lower boundary $z = 0$ with z-axis perpendicular to it. It is further assumed that cross diffusion effects may be neglected.

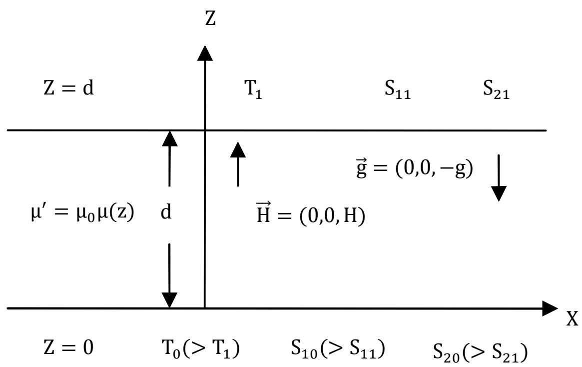

Fig.1 Physical Configuration

The basic hydrodynamic equation that governs the magnetohydrodynamic triply diffusive instability problem are given by (Prakash et al. [12], Gupta and Kaushal [5])

$$\mu(D^2 - a^2)^2 w + D^2\mu(D^2 + a^2)w + 2D\mu D(D^2 - a^2)w - \frac{p}{\sigma}(D^2 - a^2)w = Ra^2\theta - R_1a^2\phi_1 - R_2a^2\phi_2 - QD(D^2 - a^2)h_z, \tag{1}$$

$$(D^2 - a^2 - p)\theta = -w, \tag{2}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{3}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \tag{4}$$

and $\left(D^2 - a^2 - \frac{p\sigma_1}{\sigma}\right)h_z = -Dw$, (5) with $w = 0 = \theta = \phi_1 = \phi_2 = D^2w$ at $z = 0$ and $z = 1$ (both boundaries are dynamically free) (6)

or $w = 0 = \theta = \phi_1 = \phi_2 = Dw$ at $z = 0$ and $z = 1$ (both boundaries are rigid) (7)

or $w = 0 = \theta = \phi_1 = \phi_2 = Dw$ at $z = 0$ and (lower boundary is rigid)

$w = 0 = \theta = \phi_1 = \phi_2 = D^2w$ at $z = 1$ (upper boundary is dynamically free) (8)

or $w = 0 = \theta = \phi_1 = \phi_2 = D^2w$ at $z = 0$ and (lower boundary is dynamically free)

$w = 0 = \theta = \phi_1 = \phi_2 = Dw$ at $z = 1$ (upper boundary is rigid) (9)

and either $h_z = 0$ on both the boundaries if the regions outside the fluid are perfectly conducting (10)

or $Dh_z = \mp a h_z$ on the upper and lower boundary respectively if the regions outside the fluid are insulating, (11)

where z is the vertical coordinate, $D = \frac{d}{dz}$ is the differentiation along the vertical direction, a^2 is the square of the wave number, $\sigma > 0$ is the Prandtl number, $\sigma_1 > 0$ is the Magnetic Prandtl number, $\tau_1 > 0$ and $\tau_2 > 0$ are the Lewis numbers for the two concentration components respectively, $R > 0$ is the Rayleigh number, $R_1 > 0$and $R_2 > 0$ are the concentration Rayleigh numbers for the concentration components S_1 and S_2 respectively, $Q > 0$ is the Chandrasekhar number, $p = p_r + ip_i$ is the complex growth rate, w is the vertical velocity, θ is the temperature, ϕ_1 and ϕ_2 are the concentration of two components S_1 and S_2 respectively and $\mu' = \mu_0\mu(z)$ where μ_0 is constant having the dimensions of viscosity and $\mu(z)$ is twice continuously differentiable function of z and is such that the ratio of the viscosities at the top and bottom boundaries is small (Stengel[18]). It may further be noted that equations (1)-(5) describe an eigen value problem for p and govern magnetohydrodynamic triply diffusive convection with variable viscosity for the boundary conditions (6)-(11).

Now we prove the following theorem:

Theorem: If $(w, \theta, \phi_1, \phi_2, h_z, p)$, $p = p_r + ip_i$, $p_r \geq 0$, $p_i \neq 0$ $R > 0, R_1 > 0, R_2 > 0$ is a non-trivial solution of equations (1)-(5) together with the boundary conditions (6)-(11) then $|p| < max\left[\sqrt{(R_1 + R_2)\sigma},\ \ Q\sigma\right]$.

Proof: Equation (1) can further be simplified as

$$D(\mu D^3w + D\mu D^2w - 2a^2\mu Dw) + a^4\mu w - \frac{p}{\sigma}(D^2 - a^2)w + a^2(D^2\mu)w = Ra^2\theta - R_1a^2\phi_1 - R_2a^2\phi_2 - QD(D^2 - a^2)h_z. \quad (12)$$

Multiplying both sides of equation (12) by w^*(the superscript $*$ henceforth denotes complex conjugation), integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^* D(\mu D^3 w + D\mu D^2 w - 2a^2\mu Dw)dz + a^4\int_0^1 \mu\,|w|^2 dz - \frac{p}{\sigma}\int_0^1 w^*(D^2-a^2)wdz + a^2\int_0^1 w^* D^2\mu wdz = Ra^2\int_0^1 w^*\theta dz - R_1a^2\int_0^1 w^*\phi_1\, dz - R_2a^2\int_0^1 w^*\phi_2\, dz - Q\int_0^1 w^* D(D^2-a^2)h_z dz\,. \tag{13}$$

Making use of equations (2)-(5), we can write

$$\int_0^1 w^* D(\mu D^3 w + D\mu D^2 w - 2a^2\mu Dw)dz + a^4\int_0^1 \mu\,|w|^2 dz - \frac{p}{\sigma}\int_0^1 w^*(D^2-a^2)wdz + a^2\int_0^1 w^* D^2\mu wdz = -Ra^2\int_0^1 \theta(D^2-a^2-p^*)\theta^*\, dz + R_1a^2\tau_1\int_0^1 \phi_1\left(D^2-a^2-\frac{p^*}{\tau_1}\right)\phi_1^*\, dz + R_2a^2\tau_2\int_0^1 \phi_2\left(D^2-a^2-\frac{p^*}{\tau_2}\right)\phi_2^*\, dz - Q\int_0^1\left(D^2-a^2-\frac{p^*\sigma_1}{\sigma}\right)h_z^*(D^2-a^2)h_z dz\,. \tag{14}$$

Integrating the various terms, by parts, for an appropriate number of times and making use of the boundary conditions (6)-(11), we get

$$\int_0^1 \mu\,(|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)dz + \frac{p}{\sigma}\int_0^1(|Dw|^2 + a^2|w|^2)\, dz + a^2\int_0^1 D^2\mu\,|w|^2 dz + Q\int_0^1|(D^2-a^2)h_z|^2 dz = Ra^2\int_0^1(|D\theta|^2 + a^2|\theta|^2 + p^*|\theta|^2)\, dz - R_1a^2\tau_1\int_0^1(|D\phi_1|^2 + a^2|\phi_1|^2 + \frac{p^*}{\tau_1}|\phi_1|^2)\, dz - R_2a^2\tau_2\int_0^1(|D\phi_2|^2 + a^2|\phi_2|^2 + \frac{p^*}{\tau_2}|\phi_2|^2)\, dz - \frac{Qp^*\sigma_1}{\sigma}\left[a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1(|Dh_z|^2 + a^2|h_z|^2)dz\right] \tag{15}$$

Equating the imaginary parts of both sides of equation (15) and cancelling $p_i(\neq 0)$throughout from the imaginary parts, we have

$$\frac{1}{\sigma}\int_0^1(|Dw|^2 + a^2|w|^2)\, dz = -Ra^2\int_0^1|\theta|^2\, dz + R_1a^2\int_0^1|\phi_1|^2\, dz + R_2a^2\int_0^1|\phi_2|^2\, dz + \frac{Q\sigma_1}{\sigma}\left[a\{(|h_z|^2)_0 + (|h_z|^2)_1\} + \int_0^1(|Dh_z|^2 + a^2|h_z|^2)dz\right]. \tag{16}$$

Now, multiplying equation (3) by its complex conjugate, integrating the resulting equation over the vertical range of z for an appropriate number of times and utilizing the boundary conditions on ϕ_1, we have

$$\int_0^1(|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2)dz + \frac{2p_r}{\tau_1}\int_0^1(|D\phi_1|^2 + a^2|\phi_1|^2)\, dz + \frac{|p|^2}{{\tau_1}^2}\int_0^1|\phi_1|^2\, dz = \frac{1}{{\tau_1}^2}\int_0^1|w|^2\, dz. \tag{17}$$

Since $p_r \geq 0$, we have from equation (17), that

$$\int_0^1|\phi_1|^2\, dz \leq \frac{1}{|p|^2}\int_0^1|w|^2\, dz\,. \tag{18}$$

In the same manner by using (4), we obtain

$$\int_0^1|\phi_2|^2\, dz \leq \frac{1}{|p|^2}\int_0^1|w|^2\, dz\,. \tag{19}$$

Multiplying equation (5) by its complex conjugate, integrating the resulting equation over the vertical range of z for an appropriate number of times and utilizing the boundary conditions on h_z , we have

$$\int_0^1|(D^2-a^2)h_z|^2dz+\frac{2p_r\sigma_1}{\sigma}\left[a\{(|h_z|^2)_0+(|h_z|^2)_1\}+\int_0^1(|Dh_z|^2+\ a^2|h_z|^2)dz\right]+\frac{|p|^2{\sigma_1}^2}{\sigma^2}\int_0^1|h_z|^2dz=\int_0^1|Dw|^2dz \qquad (20)$$

which implies that

$$\int_0^1|h_z|^2\,dz\le\frac{\sigma^2}{|p|^2{\sigma_1}^2}\int_0^1|Dw|^2dz \qquad (21)$$

and $\int_0^1|(D^2-a^2)h_z|^2dz\le\int_0^1|Dw|^2dz$ (22)

respectively.

Now using inequalities (21) and (22), we obtain

$$a\{(|h_z|^2)_0+(|h_z|^2)_1\}+\int_0^1(|Dh_z|^2+a^2|h_z|^2)\,dz\le-\int_0^1 {h_z}^*(D^2-a^2)h_zdz\,,$$

$$\le\left|\int_0^1 {h_z}^*(D^2-a^2)h_zdz\right|\le\left[\int_0^1|h_z|^2\,dz\right]^{\frac{1}{2}}\left[\int_0^1|(D^2-a^2)h_z|^2dz\right]^{\frac{1}{2}}\le\frac{\sigma}{|p|\sigma_1}\int_0^1|Dw|^2dz \qquad (23)$$

Now utilizing inequalities (18), (19) and (23) in equation (16), we get

$$\left[\frac{1}{\sigma}-\frac{Q}{|p|}\right]\int_0^1|Dw|^2\,dz+a^2\left[\frac{1}{\sigma}-\frac{(R_1+R_2)}{|p|^2}\right]\int_0^1|w|^2\,dz+Ra^2\int_0^1|\theta|^2\,dz<0\,, \qquad (24)$$

which clearly implies that

$$|p|<max\left\{\sqrt{(R_1+R_2)\sigma},\ Q\sigma\ \right\}. \qquad (25)$$

The above theorem may be stated in an equivalent form as: the complex growth rate of an arbitrary, neutral or unstable oscillatory perturbation of growing amplitude in a magnetohydrodynamic triply diffusive fluid layer (with variable viscosity) heated from below, must lie inside a semicircle in the right half of the (p_r, p_i) – plane whose centre is at the origin and radius equals $max\{\sqrt{(R_1+R_2)\sigma}, Q\sigma\}$. Further, it is proved that this result is uniformly valid for quite general nature of the bounding surfaces.

Special Cases: The following results may be obtained from above theorem as special cases:

i) For Magnetohydrodynamic Rayleigh-Benard convection with variable viscosity $(R_1=0=R_2, Q>0)$, $|p|<Q\sigma$.
ii) For Thermohaline convection of Veronis type [2] with variable viscosity $(R_1>0, R_2=0=Q)$, $|p|<\sqrt{R_1\sigma}$.
iii) For Magnetohydrodynamic Thermohaline convection of Veronis type [2] with variable viscosity $(R_1>0, R_2=0, Q>0)$, $|p|<max\{\sqrt{R_1\sigma}, Q\sigma\}$.
iv) For Magnetohydrodynamic triply diffusive convection analogous to Stern type[25] with variable viscosity $(R<0, R_1<0, R_2<0, Q>0)$, $|p|<max\left\{\sqrt{|R|\sigma}, Q\sigma\right\}$

Proof: Putting $R_1=-|R_1|$ and $R_2=-|R_2|$ in equation (1), and adopting the same procedure as is used to prove above Theorem, we obtain the desire result.

v) For Thermohaline convection of Stern type [25] with variable viscosity ($R < 0, R_1 < 0, R_2 = 0 = Q$), $|p| < \sqrt{|R|\sigma}$.
vi) For Magnetohydrodynamic Thermohaline convection of Stern type [25] with variable viscosity ($R < 0, R_1 < 0, R_2 = 0, Q > 0$), $|p| < max\left\{\sqrt{|R|\sigma}, Q\sigma\right\}$.

Conclusion

A linear stability analysis is used to derive the upper bounds for complex growth rates in magnetohydrodynamic triply diffusive convection problem with variable viscosity. This analysis is important especially when both the boundaries are not dynamically free so that exact solutions in the closed form are not obtainable. Further, the results so obtained are uniformly valid for quite general nature of the bounding surfaces.

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Rigidly fixed vibrations of functionally graded viscothermoelastic sphere

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Abstract

The present study is based on free vibrations of in – homogenous viscothermoelastic hollow sphere. The material is assumed to be graded in radial direction with a simple power law. Matrix Fröbenious method of extended power series is employed to obtain the analytical solution for displacement and temperature. Numerical iteration technique has been used by MATLAB software tools. The computer simulated results for polymethyl methecrylate material in respect of natural frequencies, thermoelastic damping have been presented graphically.

Key Words: Functionally graded; Rigidly fixed; Vibrations; Fröbenius; Thermoelastic Damping.

1. Introduction

According to Schaflauch et al. [1] the great achievements have been made by the authors [2 – 5] to obtain general solution of the vibration problems for an isotropic sphere. Ding et al. [6] obtained the eigen frequencies of an anisotropic elastic sphere. Neuringer [7] developed the procedure of Fröbenius method when the roots of indicial equation are complex. Othman et al. [8] studied the plane waves in viscothermoelasticity in the context of generalized thermoelasticity by two relaxation times. Sharma et al. [9 – 10] studied the free vibration analysis of homogenous isotropic viscothermoelastic solid sphere and hollow sphere by using matrix Fröbenius method. Keles and Tutuncu [11] investigated the free and forced vibrations of functionally graded elastic spheres and cylinders. Dhaliwal and Singh [12] have given a detailed look to such types of problems.

The purpose of present paper is to study the exact vibration analysis of inhomogeneous isotropic, viscothermoelastic sphere subjected to rigidly fixed, thermally insulated conditions. The problem has been modeled with the help of non-classical theories of thermoelasticity developed by Lord and Shulman [13] and Green and Lindsay [14]. The secular equations have been solved with the help of MATLAB software tools for different modes of vibrations The computer simulated results in respect of natural frequencies and thermoelastic damping are shown graphically.

2. Formulation of Problem

Consider a thick walled thermally conducting viscothermoelastic hollow sphere of inner radius a and outer radius la initially at uniform temperature T_0 in the undisturbed state. For plane strain problem, the components of displacement in spherical coordinated $(r,\ \theta,\ \phi)$ system are expressed as $u_\theta = u_\phi = 0$ and $u_r = u(r,\ t)$ respectively. The basic governing equations are given by [12]:

$$\sigma_{ij,j} = \rho \ddot{u} \tag{1}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial T}{\partial r}\right) - \rho C_e(\dot{T} + t_0 \ddot{T}) = T_0 \beta^* (\dot{e} + t_0 \delta_{1k} \ddot{e}) \tag{2}$$

where

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + 2\lambda \frac{u}{r} - \beta^*\left(T + t_1 \delta_{2k} \dot{T}\right) \tag{3}$$

$$\sigma_{\theta\theta} = (\lambda + 2\mu)\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \beta^*\left(T + t_1 \delta_{2k} \dot{T}\right) \tag{4}$$

Here $\vec{u} = u(r,\ t)$ is the displacement vector; $T(r,\ t)$ is the temperature; σ_{ij} and e_{ij}, $(i, j = r,\ t)$ are stress and strain components, respectively; ρ is mass density; C_e is the specific heat at constant strain; K is the thermal conductivity; t_0 and t_1 are the thermal relaxation times and β^* is the viscothermoelastic coupling constant. The quantity δ_{ik} ,$(i = 1,\ 2)$, is the Kronecker's delta in which $k = 1$ corresponds to Lord-Shulman (LS) theory and $k = 2$ represents Green-Lindsay (GL) theory. The superposed dots represent time differentiation.

We consider the material is isotropic and functionally graded in the sense that the modulus of elasticity, thermal conductivity and density vary with the radial coordinate according as $\lambda = \lambda_0 (r)^\beta$, $\mu = \mu_0 (r)^\beta$, $\beta^* = \beta_0^* (r)^\beta$, $\rho = \rho_e (r)^\beta$, $K = K_0 (r)^\beta$,

where the exponent β essentially represents the degree of non-homogeneity.

The material parameter have been defined as

$$\lambda_0 = \lambda_e\left(1 + \alpha_0 \frac{\partial}{\partial t}\right), \quad \mu_0 = \mu_e\left(1 + \alpha_1 \frac{\partial}{\partial t}\right), \quad \beta_0^* = \beta_e\left(1 + \beta_0 \frac{\partial}{\partial t}\right)$$

$\beta_e = (3\lambda_e + 2\mu_e)\alpha_T$, $\beta_0 = (((3\lambda_e \alpha_0 + 2\mu_e \alpha_1)\alpha_T)/\beta_e)$ are the viscothermoelastic parameters and viscothermoelastic coupling parameters. The quantities α_0 , α_1 are the thermal relaxation times; λ_e, μ_e are Lame's parameters and α_T is the coefficient of linear thermal expansion of the material.

Boundary Conditions

We consider the exact analysis of non-homogenous hollow sphere which is subjected to rigidly fixed, thermally insulated conditions at inner radius $r = a$ and $r = la$. Mathematically, this provides us:

Thermally Insulated boundary conditions $\quad u = 0\,,\quad T_{,r} = 0\,,\quad$ at $\quad r = a\,,\quad la\,.$ (5)

3. Solution of the problem

In order to facilitate the solution we introduce the following non – dimensional quantities

$$U = \frac{u}{a}\,,\quad X = \frac{r}{a}\,,\quad \tau = \frac{c_1 t}{a}\,,\quad \theta = \frac{T}{T_0}\,,\quad \omega^* = \frac{C_e(\lambda_e + 2\mu_e)}{K_0}\,,\quad \varepsilon_T = \frac{T_0\beta_e^2}{\rho_e C_e(\lambda_e + 2\mu_e)}\,,$$

$$\bar{\varepsilon} = \frac{T_0\beta_e}{(\lambda_e + 2\mu_e)}\,,\quad \delta_0 = \hat{\alpha}_0 + 2\delta^2(\hat{\alpha}_1 - \hat{\alpha}_0),\quad c_1^2 = \frac{(\lambda_e + 2\mu_e)}{\rho_e}\,,\quad c_2^2 = \frac{\mu_e}{\rho_e}\,,$$

$$\delta^2 = \frac{c_2^2}{c_1^2}\,,\quad \tau_0 = \frac{c_1}{a}t_0\,,\quad \tau_0' = \frac{c_1}{a}t_0\,,\quad \tau_1 = \frac{c_1}{a}t_1\,,\quad \hat{\alpha}_0 = \frac{c_1}{a}\alpha_0\,,\quad \hat{\alpha}_1 = \frac{c_1}{a}\alpha_1\,,$$

$$\hat{\beta}_0 = \frac{c_1}{a}\beta_0\,,\quad \Omega^* = \frac{a\,\omega^*}{c_1},\quad \tau_{XX} = \frac{\sigma_{rr}}{\rho_e c_1^2}\,,\quad \tau_{\theta\theta} = \frac{\sigma_{\theta\theta}}{\rho_e c_1^2}\,, \tag{6}$$

Using quantities (6) and (3) - (4) in equations (1) and (2) and simplifying we get

$$\left(1+\delta_0\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2 U}{\partial X^2} + \frac{m_1}{X}\frac{\partial U}{\partial X} + \frac{\hat{m}_2}{X^2}U\right) - \bar{\varepsilon}\left(1+\hat{\beta}_0\frac{\partial}{\partial\tau}\right)\left(1+\tau_1\delta_{2k}\frac{\partial}{\partial\tau}\right)\left(\frac{\partial\theta}{\partial X} + \frac{\beta}{X}\theta\right) = \frac{\partial^2 U}{\partial\tau^2} \tag{7}$$

$$\frac{\partial^2\theta}{\partial X^2} + \frac{m_1}{X}\frac{\partial\theta}{\partial X} - \Omega^*\left(\frac{\partial}{\partial\tau} + \tau_0\frac{\partial^2}{\partial\tau^2}\right)\theta = \varepsilon_T\frac{\Omega^*}{\bar{\varepsilon}}\left(1+\hat{\beta}_0\frac{\partial}{\partial\tau}\right)\left(\frac{\partial}{\partial\tau} + \tau_0'\delta_{1k}\frac{\partial^2}{\partial\tau^2}\right)\left(\frac{\partial U}{\partial X} + \frac{2}{X}U\right) \tag{8}$$

where $\quad m_1 = \beta + 2\,,\qquad \hat{m}_2 = 2\left(\dfrac{(1-2\delta^2)\left(1+\hat{\alpha}_0(\partial/\partial\tau)\right)\beta}{\left(1+\hat{\delta}_0(\partial/\partial\tau)\right)} - 1\right)$

4. Introduction of time harmonics and transformation

We consider time harmonic vibrations and transformation as

$$U = X^{-\frac{1+\beta}{2}}\,\overline{U}\,\exp(-i\Omega\tau) \tag{9}$$

$$\theta = X^{-\frac{1+\beta}{2}}\,\Theta\,\exp(-i\Omega\tau)$$

Using equation (9) in equations (7) and (8) and simplifying we get

$$\begin{pmatrix} \nabla^2+\left(\dfrac{i\Omega}{\widetilde{\delta}_0}-\dfrac{n^2}{X^2}\right) & a^*\left(\dfrac{d}{dX}+\dfrac{(\beta-1)}{2X}\right) \\ -b^*\left(\dfrac{d}{dX}+\dfrac{3-\beta}{2X}\right) & \nabla^2+\left(\Omega^*\Omega^2\widetilde{\tau}_0-\left(\dfrac{1+\beta}{2}\right)^2\dfrac{1}{X^2}\right) \end{pmatrix}\begin{pmatrix} \overline{U} \\ \Theta \end{pmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

where $a^*=\dfrac{i\Omega\bar{\varepsilon}\widetilde{\beta}_0\,\widetilde{\tau}_1}{\widetilde{\delta}_0}$, $b^*=i\Omega^3 m_4\widetilde{\beta}_0\widetilde{\tau}_0'$, $m_2=2\left(\dfrac{\beta\widetilde{\alpha}_0(1-2\delta^2)}{\widetilde{\delta}_0}-1\right)$, $m_4=\dfrac{\varepsilon_T\Omega^*}{\bar{\varepsilon}}$,

$\widetilde{\alpha}_0=i\Omega^{-1}+\hat{\alpha}_0$, $\widetilde{\alpha}_1=i\Omega^{-1}+\hat{\alpha}_1$, $\widetilde{\beta}_0=i\Omega^{-1}+\hat{\beta}_0$, $\delta_0=i\Omega^{-1}+\delta_0$, $\widetilde{\tau}_1=i\Omega^{-1}+\tau_1\,\delta_{2k}$,

$\widetilde{\tau}_0=i\Omega^{-1}+\tau_0$, $\widetilde{\tau}_0'=i\Omega^{-1}+\tau_0',\delta_{1k}$, $n^2=\left(\dfrac{1+\beta}{2}\right)^2-m_2$, $\nabla^2=\dfrac{1}{X}\dfrac{d}{dX}\left(X\dfrac{d}{dX}\right)$

5. Solution by applying Matrix Fröbenius Method

Clearly, the dimensional domain of consideration $a\le r\le la$ corresponds to $1\le X\le l$ in non-dimensional form. In order to apply the matrix Fröbenius method to solve equations (10) we look for power series of the type

$$Z=\sum_{k=0}^{\infty} Z_k\,X^{p+k} \tag{11}$$

where $Z=\left[\overline{U}\quad\Theta\right]'$ and $Z_k=\left[A_k\quad B_k\right]'$, Here p is the eigen value and $A_k\,,B_k$ are unknown coefficients to be determined.

Substituting the solution (11) in equations (10) we get following system of equations

$$\sum_{k=0}^{\infty}\left[H_1(p+k)X^{-2}+H_2(p+k)X^{-1}+H\right]X^{p+k}Z_k=0 \tag{12}$$

where $H_1(p+k)=\begin{bmatrix}(p+k)^2-n^2) & 0 \\ 0 & \left((p+k)^2-\left(\dfrac{1+\beta}{2}\right)^2\right)\end{bmatrix}$,

$$H_2(p+k)=\begin{bmatrix} 0 & a^*\left(p+k-\dfrac{\beta-1}{2}\right) \\ -b^*\left(p+k+\dfrac{3-\beta}{2}\right) & 0 \end{bmatrix}\quad,\quad H=\begin{bmatrix}\dfrac{i\Omega}{\delta_0} & 0 \\ 0 & \Omega^*\Omega^2\widetilde{\tau}_0\end{bmatrix}$$

Equating to zero, the coefficients of lowest power of X (*i.e.* X^{p-2}) in equation (12), we obtain

$$H_1(p)Z_0 = 0 \tag{13}$$

The system of equations (13) will have a non-trivial solution iff $\left|H_1(p)\right| = 0$, which leads to the indicial equation

$$\left[p^2 - n^2\right]\left[p^2 - \left(\frac{1+\beta}{2}\right)^2\right] = 0 \tag{14}$$

The roots of above equations are given as

$$p_1 = n \quad , \quad p_2 = -n \quad p_3 = \left(\frac{1+\beta}{2}\right) , \qquad p_4 = -\left(\frac{1+\beta}{2}\right) \tag{15}$$

Clearly, these roots satisfy the property $p_2 = -p_1$, $p_4 = -p_3$. Here the roots p_i $(i = 1\ ,\ 2)$ are complex and the roots p_i $(i = 3\ ,\ 4)$ being real. Thus, in the former case the leading terms in the series solution (11) are of the type

$$\left[A_0 \quad B_0\right] X^p = \left\{A_0 \quad B_0\right\} X^{p_R + i\,p_I} = \left\{A_0 \quad B_0\right\} X^{p_R} \left\{\cos\,(p_I \log X) + i \sin\,(p_I \log X)\right\}$$

In order to obtain two independent real solutions, it is sufficient to use any one of the complex root and taking its real and imaginary parts see Neuringer [7].

For the choice of indicial roots, the system of equations (14) leads to:

$$A_0\left(p_j\right) = \begin{cases} 1\ ,\ j = 1\ ,\ 2 \\ 0\ ,\ j = 3\ ,\ 4 \end{cases} , \qquad B_0\left(p_j\right) = \begin{cases} 0\ ,\ j = 1\ ,\ 2 \\ 1\ ,\ j = 3\ ,\ 4 \end{cases} \tag{16}$$

Again equating to zero the coefficients of next lowest degree term X^{p-1} in equation (13), we obtain

$$H_1(p_j + 1)Z_1 + H_2(p_j)Z_0 = 0 \tag{17}$$

The equation (18) on simplification gives us a solution

$$Z_1 = -D_1 Z_0 \tag{18}$$

where

$$D_1 = \begin{bmatrix} 0 & d_{12}^1(p_j) \\ d_{21}^1(p_j) & 0 \end{bmatrix} \text{ and } d_{12}^1(p_j) = \frac{a^*\left(p_j - \dfrac{3-\beta}{2}\right)}{(p_j+1)^2 - n^2}, d_{21}^1(p_j) = \frac{-b^*\left(p_j + 1 + \dfrac{3-\beta}{2}\right)}{(p_j+1)^2 - \left(\dfrac{1+\beta}{2}\right)^2}$$

Now equating the coefficients of like powers of X^{p+k} equal to zero, we obtain the recurrence relation:

$$H_1(p_j+k+2)Z_{k+2}+H_2(p_j+k+1)Z_{k+1}+H\,Z_k=0\ ,\quad k=0,1,2,3,... \tag{19}$$

On simplification the equation (19) implies that

$$Z_{k+2}=-\begin{bmatrix}0 & H_{12}^k(p_j)\\ H_{21}^k(p_j) & 0\end{bmatrix}Z_{k+1}-\begin{bmatrix}H_{11}^k(p_j) & 0\\ 0 & H_{22}^k(p_j)\end{bmatrix}Z_k\quad ,\quad k=0,1,2,3,... \tag{20}$$

where $H_{11}^k(p_j)=\dfrac{i\Omega}{\delta_0\{(p_j+k+2)^2-n^2\}}$, $H_{12}^k(p_j)=\dfrac{a^*\left(p_j+k+1-\frac{\beta-1}{2}\right)}{(p_j+k+2)^2-n^2}$

$$H_{21}^k(p_j)=\frac{-b^*\left(p_j+k+1+\frac{3-\beta}{2}\right)}{(p_j+k+2)^2-\left(\frac{1+\beta}{2}\right)^2}\quad ,\quad H_{22}^k(p_j)=\frac{m_3\Omega^2\tilde{\tau}_0}{(p_j+k+2)^2-\left(\frac{1+\beta}{2}\right)^2}$$

For $k=0$, the equation (20) upon simplifications provides us

$$Z_2=D_2Z_0 \tag{21}$$

where

$$D_2=\begin{bmatrix}d_{11}^2(p_j) & 0\\ 0 & d_{22}^2(p_j)\end{bmatrix}\ \text{and}\ d_{11}^2(p_j)=\begin{Bmatrix}H_{12}^0(p_j)d_{21}^1(p_j)\\ -H_{11}^0(p_j)\end{Bmatrix},\ d_{22}^2(p_j)=\begin{Bmatrix}H_{12}^0(p_j)d_{12}^1(p_j)\\ -H_{22}^0(p_j)\end{Bmatrix}$$

and putting $k=1, 2, 3, 4$ ....so on. Continuing in this manner it can be easily shown that the matrices $D_{2k}(p_j)$ have similar form as that of $H_1(p+k)$ and the matrices $D_{2k+1}(p_j)$ are alike $H_2(p+k)$. Thus, in general, we have

$$Z_{2k}(p_j)=D_{2k}(p_j)Z_0\ ,\ k=1\ \ 2,3\ ...\ , \tag{22}$$

$$Z_{2k+1}(p_j)=-D_{2k+1}(p_j)Z_0\ ,\ k=1,\ 2,3\ ... \tag{23}$$

where $D_{2k}(p_j)=\begin{bmatrix}d_{11}^{2k}(p_j) & 0\\ 0 & d_{22}^{2k}(p_j)\end{bmatrix}$, $D_{2k+1}(p_j)=\begin{bmatrix}0 & d_{12}^{2k+1}(p_j)\\ d_{21}^{2k+1}(p_j) & 0\end{bmatrix}$

$$d_{11}^{2k}(p_j)=\{H_{12}^{2k-2}(p_j)d_{21}^{2k-1}(p_j)-H_{11}^{2k-2}(p_j)d_{11}^{2k-2}(p_j)\}$$
$$d_{22}^{2k}(p_j)=\{H_{21}^{2k-2}(p_j)d_{12}^{2k-1}(p_j)-H_{22}^{2k-2}(p_j)d_{22}^{2k-2}(p_j)\}$$

$$d_{12}^{2k+1}(p_j) = \left\{ -H_{12}^{2k-1}(p_j) d_{22}^{2k}(p_j) + H_{11}^{2k}(p_j) d_{12}^{2k-1}(p_j) \right\}$$
$$d_{21}^{2k+1}(p_j) = \left\{ -H_{21}^{2k-1}(p_j) d_{11}^{2k}(p_j) + H_{22}^{2k-1}(p_j) d_{21}^{2k-1}(p_j) \right\}$$
where $d_{11}^{0}(p_j) = 1 = d_{22}^{0}(p_j) \quad ; \quad j = 1, 2, 3, 4.$

6. Convergence Analysis

From equation (22) – (23), it can also be shown that

$$D_{2k}(p_j) \approx O\left(k^{-1}\right) E^{*} \quad , \quad D_{2k+1}(p_j) \approx O\left(k^{-1}\right) E^{**} \tag{24}$$

where $E^{*} = \begin{bmatrix} a^{*} & 0 \\ 0 & -b^{*} \end{bmatrix}$ and $E^{**} = \begin{bmatrix} 0 & a^{*} \\ -b^{*} & 0 \end{bmatrix}$

Now according to Cullen [15], a matrix sequence $\{A_k\}$ in the complex filed converges, $(\lim_{k\to\infty} A_k = A)$, if each of the k^2 component sequence is convergent. Upon utilizing the above stated fact, we see that both the matrices $D_{2k}(p_j) \to 0$ and $D_{2k+1}(p_j) \to 0$, as $k \to \infty$. This implies that the series (11) is absolutely and uniformly convergent having infinite radius of convergence and the derived series is analytic functions and hence can be differentiated term by term.

Thus the series solution (11) becomes

$$Z = (I - D_1 X + D_2 X^2 - D_3 X^3 + D_4 X^4 - D_5 X^5 + \ldots \quad) X^{p_j} Z_0 \tag{25}$$

where I is an identity matrix of order two and matrices $D_i (i = 1, 2, 3 \ldots)$ have been defined above

7. Formal Solution to obtain displacement and temperature gradient

In the light of the above discussion, the series solution (26) with help of equations (5) via equation (9) displacement, temperature and temperature gradient are written as:

$$U(X, \tau) = \sum_{k=0}^{\infty} \begin{bmatrix} E_1 d_{11}^{2k}(p_1) X^{p_1} + E_2 d_{11}^{2k}(p_2) X^{p_2} - E_3 d_{12}^{2k+1}(p_3) X^{1+p_3} \\ -E_4 d_{12}^{2k+1}(p_4) X^{1+p_4} \end{bmatrix} X^{2k - \frac{\beta+1}{2}} \exp(-i\Omega\tau)$$

$$\theta_{,X} = \sum_{k=0}^{\infty} \begin{bmatrix} -\sum_{j=1}^{2} \left(2k + \frac{1-\beta}{2} + p_j \right) E_j d_{21}^{2k+1}(p_j) \\ + \sum_{j=3}^{4} \left(2k - \frac{1+\beta}{2} + p_j \right) \frac{1}{X} E_j d_{22}^{2k}(p_j) \end{bmatrix} X^{2k + p_j - \frac{1+\beta}{2}} e^{-i\Omega\tau} \tag{26}$$

where $E_j \ (j =, 1, 2, 3, 4)$ are arbitrary constants to be evaluated.

8. Secular dispersion equations

We assume the viscothermoelastic sphere is subjected to rigidly fixed and thermally insulated conditions (5) at its surfaces ($X = 1\ ,\ l$). The system will have a nontrivial solution if and only if the determinant of the coefficients $E_j\ \ (j = 1,\ 2,\ 3,\ 4)$ vanishes. This requirement of nontrivial solution leads to following dispersion equations as discussed below:

Case I: For $k = 0$. In this case the secular equations are obtained as:

$$\det\,(m^*_{ij}) = 0\ , \quad (i, j = 1,\ 2,\ 3,\ 4) \tag{27}$$

where the elements m^*_{ij} have been defines as below:

$$\begin{aligned}
&m^*_{1j} = 1\ ;\ \ j = 1\ ,2. \quad ; \quad m^*_{1j} = -d^1_{12}(p_j)\ ;\ \ j = 3\ ,4.\\
&m^*_{2j} = \ (l)^{p_j - \frac{1+\beta}{2}}\ ;\ \ j = 1\ ,2. \quad ; \quad m^*_{2j} = d^1_{12}(p_j)\ (l)^{p_j + 1 - \frac{1+\beta}{2}}\ ;\ \ j = 3\ ,4.\\
&m^*_{3j} = -\left(\frac{1-\beta}{2} + p_j\right) d^1_{21}(p_j)\ ;\ j = 1\ ,2.\ ;\quad m^*_{33} = 0\ ;\quad m^*_{34} = -(1+\beta)\\
&m^*_{4j} = m^*_{3j}\,(l)^{p_j - \frac{1+\beta}{2}}\ \ ;\ j = 1\ ,2\ \ ;\ \ m^*_{43} = 0\quad ;\quad m^*_{44} = m^*_{34}\ \,(l)^{-(2+\beta)}
\end{aligned} \tag{28}$$

Case: II For $k > 0$. In this case the secular equations for Set I and Set II are obtained as:

$$\det\,(m_{ij}) = 0\ , \quad (i, j = 1,\ 2,\ 3,\ 4) \tag{29}$$

where the elements of m_{ij} are defined as below:

$$\begin{aligned}
&m_{1j} = d^{2k}_{11}\ \ ;\ \ j = 1\ ,2. \quad ; \quad m_{1j} = -d^{2k+1}_{12}(p_j)\ ;\ \ j = 3\ ,4.\\
&m_{2j} = d^{2k}_{12}(p_j)\ \ (l)^{p_j - \frac{1+\beta}{2}}\ ;\ \ j = 1\ ,2.\ ; \quad m_{2j} = -d^{2k+1}_{12}(p_j)\,(l)^{p_j + 1 - \frac{1+\beta}{2}}\ ;\ \ j = 3\ ,4.\\
&m_{3j} = -\left[\left(2k + \frac{1-\beta}{2} + p_j\right) d^{2k+1}_{21}(p_j)\right], j = 1\ ,2.\ ;\ m_{3j} = \left[\left(2k - \frac{1+\beta}{2} + p_j\right) d^{2k}_{22}(p_j)\right], j = 3\ ,4.\\
&m_{4j} = m_{3j}\,(l)^{2k + p_j - \frac{1+\beta}{2}}\ \ ;\ \ j = 1\ ,2. \quad ; \quad m_{4j} = m_{3j}\,(l)^{2k + p_j - \frac{3+\beta}{2}}\ \ ;\ \ j = 3\ ,4.
\end{aligned} \tag{30}$$

The secular dispersion equations (27) and (29) govern axisymmetric vibrations of functionally graded viscothermoelastic sphere under rigidly fixed thermally insulated conditions prevailing at its surface.

9. Numerical results and discussion

In order to illustrate the analytical development, we propose some numerical results in this section. Here numerical computations have been carried out in case of rigidly fixed thermally insulated sphere by employing fixed point iteration numerical technique with the help of MATLAB software. The polymethyl methacrylate material has been considered for numerical computations whose physical data is given below Othman et al. [8]:

$\varepsilon_T = 0.045$, $\omega^* = 1.11\times10^{11}\, s^{-1}$, $T_0 = 773\, K$, $\delta^2 = 0.333$,

$\hat{\alpha}_0 = \hat{\alpha}_1 = 0.05$, $\tau_0 = 0.02$, $\tau_1 = 0.03$, $\rho = 1190\ \ kg\ \ m^{-3}$,

$K = 0.19 W\, m^{-1} K^{-1}$, $C_e = 1400\, J\, kg^{-1} K^{-1}$, $\alpha_T = 77\times10^{-6}\ K^{-1}$

Due to the presence of dissipation term in heat conduction equation (2), the secular equations are, in general, complex transcendental equations and hence provide us complex values of the natural frequency Ω. If we write $\Omega^m = \Omega_R^m + i\Omega_I^m$, the non – dimensional frequency and dissipation factor are given by $f_v = \Omega_R^m$ and $D = \Omega_I^m$, where m is the mode number which corresponds to the roots of the transcendental equation (29). The numerical computations have been done from equation (29) by taking sufficient number of the values of Fröbenius parameter (k) in order to obtain the natural frequency f_v and dissipation factor (D) of different modes. The computer simulated natural frequency, thermoelastic damping and frequency shift have been presented graphically for viscothermoelastic (VTE), thermoelastic (TE), viscoelastic (VE) and elastic (E) spheres. Here the thermoelastic damping (Q^{-1}) and frequency shift (Ωs) are defined as [16], $Q^{-1} = 2\left|\dfrac{D}{f_v}\right|$ and $\Omega s = \left|\dfrac{f_v^M - f_v^E}{f_v^E}\right|$ respectively. Here M stands for VTE (viscothermoelastic), TE (thermoelastic), VE (viscoelastic) materials and E denotes elastic one.

Figs. 1 and 2 present variation of non – dimensional frequency (f_v) versus mode number (m) for $l = 2$, $l = 4$ and different values of grading index (β) . It is observed that the non – dimensional frequency (f_v) increases with mode number (m) for $l = 2$ and $l = 4$. The frequency increases with grading index (β) in the order $\beta = -2.0,\ 0.0,\ -5.0,\ 2.0,\ 5.0$ for $l = 2$ where as it happens in the order $\beta = 0.0,\ -2.0,\ 2.0,\ -5.0,\ 5.0$ for $l = 4$. Thus, the magnitude of the frequency (f_v) remains large for $\beta = 5$ in both the cases for all considered values of (m) in comparison to other values of (β) and it has small magnitude for $\beta = -2.0$ and $\beta = 0.0$ in case of $l = 2$ and $l = 4$, respectively. This depicts the effect of in – homogeneity parameter on the variations of non – dimensional frequency.

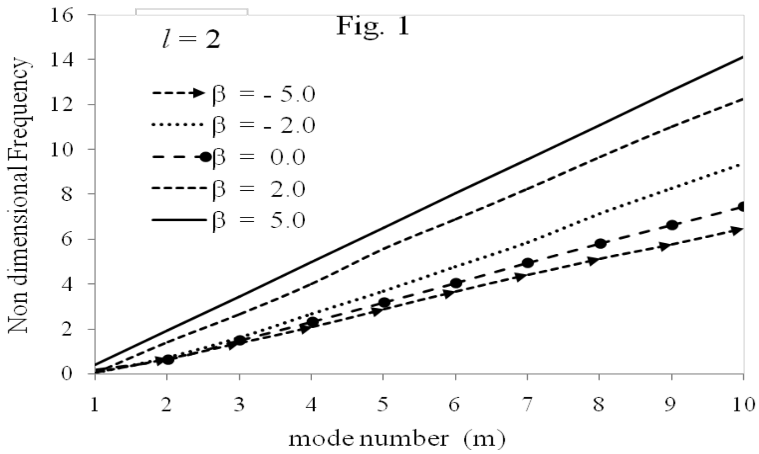

Fig. 1: Non – dimensional frequency (f_v) versus mode number (m) for different values of β and $l = 2$.

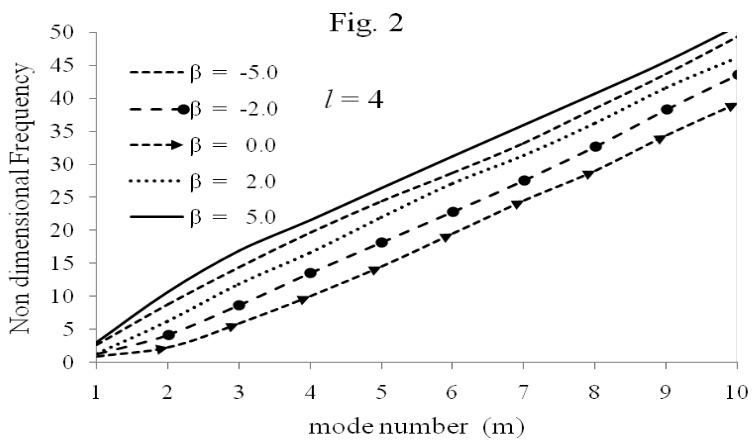

Fig. 2: Non-dimensional frequency (f_v) versus mode number (m) for different values of β and $l = 4$.

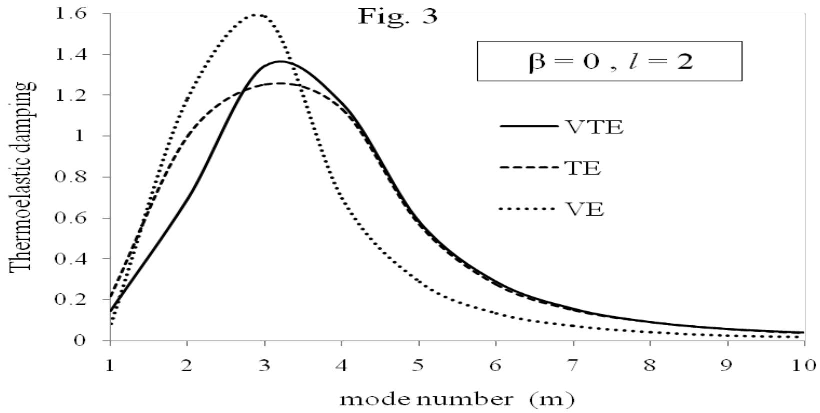

Fig. 3. Thermoelastic damping (Q^{-1}) versus mode number (m) for VTE, TE and VE $(\beta = 0\ ,\ l = 2)$.

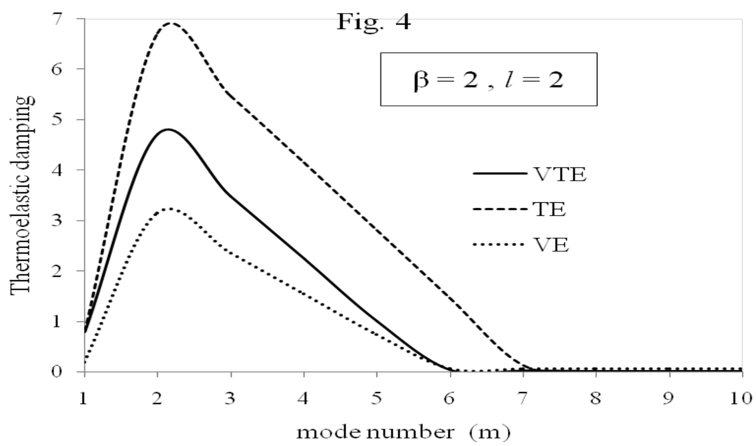

Fig. 4: Thermoelastic damping (Q^{-1}) versus mode number (m) VTE, TE and VE $(\beta = 2\ ,\ l = 2)$.

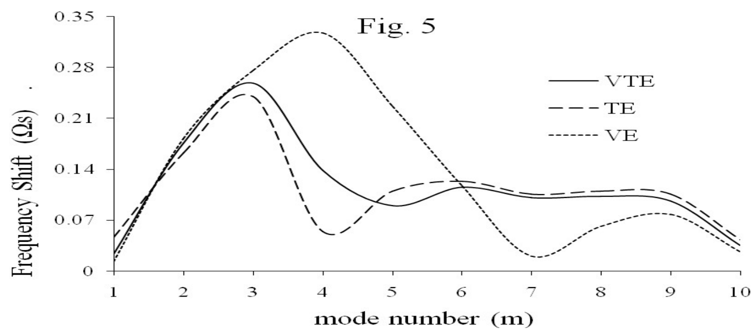

Fig. 5: Frequency shift (Ωs) versus mode number (m) for VTE, TE and VE ($\beta = 0$, $l = 2$).

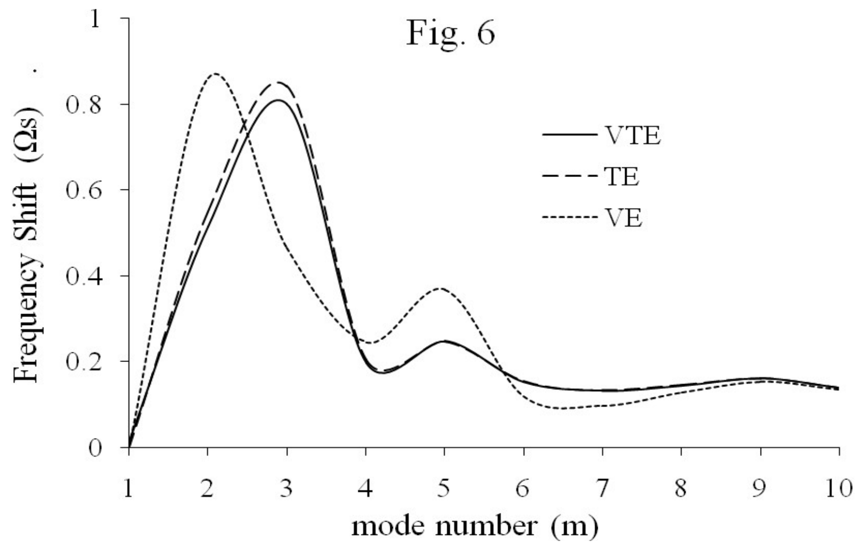

Fig. 6: Frequency shift (Ωs) versus mode number (m) for VTE, TE and VE ($\beta = 2$, $l = 2$).

Figs. 3 and 4 show the variations of thermoelastic damping (Q^{-1}) between VTE, TE and VE for $\beta = 0$, $l = 2$ and $\beta = 2$, $l = 2$. It is revealed that thermoelastic damping (Q^{-1}) profiles initially increase to attain their peak values and then decrease with increasing mode number (m) . This peak value of the quantity varies according as $Q_{TE}^{-1} < Q_{VTE}^{-1} < Q_{VE}^{-1}$ for $\beta = 0$, $l = 2$ and obeys the inequalities $Q_{VE}^{-1} < Q_{VTE}^{-1} < Q_{TE}^{-1}$ for $\beta = 2$, $l = 2$. Thus, the in – homogeneity index (β) significantly affects the existence of peak value. The peak value of Q_{VTE}^{-1} seems to be more or less the average of Q_{TE}^{-1} and Q_{VE}^{-1} in both the cases with reversed trends. The variations of frequency shift (Ωs) versus mode number (m) have been plotted in Figs. 5 and 6 for $\beta = 0$, $l = 2$ and $\beta = 2$, $l = 2$. It is noticed that frequency shift of vibrations is quite high for VE materials as compared to that for VTE and TE spheres in both the cases $\beta = 0$, $l = 2$ and $\beta = 2$, $l = 2$.

10. Conclusion

The Matrix Fröbenius method has been successfully implemented to study axisymmetric rigidly fixed vibrations of viscothermoelastic spheres. The in-homogeneity parameter significantly affects the vibration characteristics. The analytically observed relations for different cases of vibrations have been analyzed numerically for polymethyl methacrylate material. Thermoelastic damping and frequency shift may also be handled with this index to enhance the quality of the signals of different modes of vibrations. The energy loses (othermoelastic damping) can also be optimized with the help of grading index. The thermal relaxation time and thermoelastic coupling parameters have significant vibrations effect on vibration characteristics such as thermoelastic damping and frequency shift. The study may find applications in industry and medicine to control the stress distribution.

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A Survey of the work on Almost Injective Modules

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Abstract. Properties of almost injective modules and indecomposable almost self-injective modules derived by various researchers since the inception of these modules (i.e. 1989) have been surveyed. The conditions have been studied as to when a direct sum of almost injective modules is again almost injective. It is observed that for any module M to be an indecomposable almost self injective module then $End(M)$ is local.

Keywords: Almost Injective Modules, Local rings, Uniform Modules, Injective Hull.

Introduction. This is a brief survey on 'almost injective modules' that has been studied mostly by Harada and his collaborators. Harada and Tozaki in [3] defined 'almost M-projective modules' which is generalized from the concept 'M-projective modules'. Further, Baba in [2] introduced the concept 'almost M-injective modules' analogous to the concept of 'almost M-projective modules'. He generalized the Azumaya's theorem concerning to 'M-injective module' to the case of 'almost M-injective module: N is M_1- and M_2-injective module iff N is $M_1 \oplus M_2$-injective module for modules M_1 and M_2 to the case of 'almost M-injective modules'. Harada in [4] extended the theorem proved by Baba in [2]to the case of Artinian modules. Mainly, in this paper, we analysed and reproduced the results of Alahmadi and Jain [1].

Preliminiries. In this paper, we always assume that R is a ring with identity and every module is unitary right R-module. For module M, the socle and injective hull of M will be denoted by Soc(M) and E(M). $N \subset_e M$ will denote that M is an essential extension of module N. If $\mathrm{End}_R(M)$ is a local ring, we say M is an LE module.

Definition1.1. Harada in [3] defined the concept of almost M-projective modules. Let M and N be two right R-modules. Let $v: M \to {M}/{K}$ be the canonical epimorphism and $h: N \to {M}/{K}$ be any R-homomorphism. If there exist an R-homomorphism $k: N \to M$ such that the diagram 1 commutes, i.e. $vk = h$, or there exist a non-zero direct summand M_1 of M (denoted by $M_{\oplus} \supseteq M$) and an R-homomorphism k': $M_1 \to N$ such that $hk' = v$ restricted to M_1 as shown in the diagram 2 then N is called almost M-projective module.

$$
\begin{tikzcd}
& N \arrow[dl, "k"'] \arrow[d, "h"] & \\
M \arrow[r, "v"'] & M/K \arrow[r] & 0
\end{tikzcd}
$$

Diagram 1

$$
\begin{tikzcd}
M_{\oplus} \supseteq M_1 \arrow[r, "k'"] \arrow[d, "f"] & N \arrow[d, "h"] & \\
M \arrow[r, "v"'] & M/K \arrow[r] & 0
\end{tikzcd}
$$

Diagram 2

Definition1.2. Baba in [2] defined the concept 'almost M-injective modules'. M is called almost $N-$ injective module if for each submodule X of N and each homomorphism $f\colon X \to M,$ either there exists homomorphism $g\colon N \to M$ such that diagram 3 commutes or there exists homomorphism $h\colon M \to N$ such that diagram 4 commutes where N_1 is a nonzero direct summand of N, and $\pi : N \to N_1$ is a projection onto N_1.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"] & N \arrow[dl, "g"] \\
 & M &
\end{tikzcd}$$

Diagram 3

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"] & N = N_1 \oplus N_2 \arrow[d, "\pi"] \\
 & M \arrow[r, "h"] & N_1
\end{tikzcd}$$

Diagram 4

Definition 1.3. (Almost self-injective module). If an R-module M is almost $M-$injective then M is called almost self-injective module.

Definition 1.4. (Right almost self-injective ring). A ring R is called right almost self injective if it is almost self injective as a right module over itself.

Definition 1.5. (Essential extension). An R-module M with submodule N is said to be essential extension of N if for each submodule $H$of M, $H \cap N = \{0\}$ implies that $H = \{0\}$.

Definition 1.6. (Injective hull). An R-module E is called the injective hull of an module$R$–module M if E is an essential extension of M and E is an injective module.

Definition 1.7. (Uniform module). An R-module$M$ is called a uniform module if Intersection of any two non-zero submodules of M is non-zero.

Definition 1.8. (Local ring). A ring R is called local if set of non-unit elements in R forms an ideal.

Definition 1.9. (Indecomposable module). An R-module M is said to be indecomposable if it is non-zero and it cannot be written as direct sum of two non-zero submodules.

Definition 1.10. (π-injective module) . An R-module M is called quasi-continuous or π-injective if for any two submodules M_1 and M_2 of M with $M_1 \cap M_2 = 0$, each projection $\pi_i : M_1 \oplus M_2 \to M_i \; for \; i = 1,2$, can be extended to an endomorphism of M.

Theorem 1.11. [Azumaya Theorem] Let N, $M_1$and M_2 be R-modules. If N isM_1-and M_2-injective, then N is $M_1 \oplus M_2$ injective.

Baba in [2] generalized the Azumaya's above theorem to the case of almost M-injective modules as following:

Theorem1.12. Let U_k be the uniform modules of finite composition length for $k = 0,1,2 \ldots n$, then the following two conditions are equivalent:

(1) U_0 is almost $\sum_{k=1}^{n} \oplus U_k$-injective.

(2) U_0 is almost U_k- injective for every $k = 1,2 \ldots n$ and if Soc(U_0)≈Soc(U_k) ≈ Soc(U_l)(for any $k, l \in \{1,2 \ldots n\}$, $k \neq l$) then (i) U_0 is U_k-and U_l-injective or (ii) $U_k \oplus U_l$ is extending for simple module.

Harada in [4] generalized the above theorem proved by Baba [2] to the case of Artinian Modules as following:

Theorem 1.13. Let U_0 and $\{U_j, I_k\}_{j=1}^{n}{}_{k=1}^{m}$ be LE and Artinian modules such that U_0 is I_j-injective for all j and U_0 is almost U_i-injective but not for all i. Then U_0 is almost $(\sum_i \oplus U_i) \oplus (\sum_j \oplus I_j)$ injective iff $\sum_i \oplus U_i)$is an extending module.

Definition1.14. Harada in [5] introduced the concept of 'almost M-simple projective modules' and 'almost M-simple injective modules'. He introduced a little weaker condition to the definition of almost M-projective module. In the diagram 1 and 2, he take only $h : N \to M/N$ whose image is simple. If for any h in these diagrams, there exist a homomorphism $\tilde{h}$, then N is called almost M-simple projective module. Similarly in the diagram 3 and 4, he take only those $f: X \to M$ whose image is simple. If for any f in the diagram 3 and 4, either there exist $g: N \to M$ such that diagram 3 commutes or there exist $h: N_1 \to M$ such that diagram4 commutes, then N is called almost M −simple injective module. He proved that the above weaker conditions coincide with the original one when R is semi perfect ring and M and N are R-modules of finite length. He gave a criterion for an R-module M_0 to be almost M_1-projective, where R is a perfect ring and M_1 is indecomposable R-module.

We have reproduced the results of Alahmadi and Jain of [1] as following:

Lemma1.15. An indecomposable almost self-injective module is π-injective and hence uniform.

Proof. Let A and B be non zero submodules of an indecomposable almost self-injective module M such that $A \cap B = 0$. Then the projection π: $A \oplus B \to A$ can either be extended to an endomorphism of M by diagram 3 or there exists a homomorphism $g \in \mathrm{End}(M)$ such that $g\pi = i$ by diagram 4. The later implies ker (π)=0, a contradiction. So M is π-injective and hence uniform module.

Lemma 1.16. Let M be a uniform module then $E(M)$ (injective hull of M) is again uniform module.

Proof: Let $K_{1,}K_2$ be submodules of $E(M)$ with $K_1 \cap K_2 = 0$, then $M \cap K_1$, $M \cap K_2$ are submodules of M and $(M \cap K_1) \cap (M \cap K_2) = M \cap (K_1 \cap K_2) = M \cap 0 = 0. M$ being uniform module implies that$M \cap K_1 = 0$ or $M \cap K_2 = 0$. So by the definition of $E(M)$, either $K_1 = 0$ or $K_2 = 0$ which shows that $E(M)$ is a uniform module.

Theorem 1.17. By [7] A module M is almost $N -$ injective iff for any homomorphism $f \in Hom(E(N), E(M))$ such that $f(N) \nsubseteq M$, the following holds:

(i) $N = N_1 \oplus N_2$for some submodules N_1, N_2 with $N_1 \neq 0$.

(ii) f is monic on $E(N_1), E(M) = f(E(N_1)) \oplus K_1$ such that $f\big(E(N_1)\big) \cap \pi_1(M) \subseteq f(N_1)$ where $\pi_1 \colon E(M) \to f(E(N_1))$ is a projection via K_1.

(iii) $f(N_2 \cap L) \subseteq K_1$, where $L = \{x \in N \colon f(x) \in M\} = f^{-1}(M) \cap N$.

(iv) For the projection $\pi_1 \colon E(M) \to f(E(N_1))$ via K_1, there exist an isomorphism $g \colon E(N_1) \to f(E(N_1))$ such that $\pi_1(M) \subseteq g(N_1)$, g maps $E(N_1)$ onto $f(E(N_1))$ and $g(x_1) = f(x_1) + \pi_1 f(x_2)$.

Proposition 1.18. Let M and N be uniform modules. Then M is almost N-injective module if and only if for every $f \in Hom\,(E(N), E(M))$ either $f(N) \subseteq M$ or f is an isomorphism and $f^{-1}(M) \subseteq N$.

Proof. Assume M is almost N-injective module.

To prove: For every $f \in Hom\,\big(E(N), E(M)\big)$ either $f(N) \subseteq M$ or f is an isomorphism and $f^{-1}(M) \subseteq N$. .
Let $f \in Hom\,\big(E(N), E(N)\big)$ and $X = \{\, n \in N \mid f(n) \in M \,\}$ then $f|_X \colon X \to M$. Since M is almost N– injective then, either diagram 3 or the diagram 4 holds. If diagram 3 holds, then there exists $g \colon N \to M$ such that $f|_X = g|_X$.

Claim $M \cap (g - f)(N) = 0$.

Assume that $m \in M \cap (g-f)(N)$ that imply $m \in M$and $m \in (g-f)(N)$ such that $m = (g - f)(n)$, for some $n \in N$.Then $f(n) = g(n) - m \in M$.(because $m \in M$ and $g : N \to M$such that $g(n) \in N$ for all $n \in N$ that implies$g(n) - m \in M$, which implies $n \in X$. So $m = g(n) - f(n) = 0$ (because $f|_X = g|_X$).

But $M \subseteq e\ E(M)$. Hence $(g-f)(N) = 0$ .That is $f(N) \subseteq M$.

If diagram 4 holds, then there exists $h: M \to N$ such that $hof(x) = I(x)$ which shows f is one-one. Since M and N are uniform modules then by lemma 1.16, $E(M)$ and $E(N)$ are uniform which implies that we cannot decompose $M, N, E(N)$ and $E(M)$. Hence, by theorem 1.17, f is an isomorphism. Clearly $h|_{f(X)} = f^{-1}|_{f(X)}$.

Again claiming $M \cap (f^{-1} - h)(M) = 0$. Let $m' \in M \cap (f^{-1} - h)(M)$ such that there exist $n' \in N, n' = (f^{-1} - h)(m')$for some $m' \in M$then $f^{-1}(m') = h(m') + n' \in N$. Apply f to both sides, we get$m' = ff(m') = f(h(m') + n')$ which implies $m' \in f(X)$. So $n' = (f^{-1} - h)(m') = 0$ because $h|_{f(X)} = f^{-1}|_{f(X)}$ and $m' \in f(X)$. Hence our claim is true. Since $N \subseteq_e E(N)$, $(f^{-1} - h)(M) = 0$. That means $f^{-1}(M) = h(M) \subseteq N$. The converse is clear.

Lemma 1.19. Let R be a ring with unity. Let $End(R_R)$ denote the ring of endomorphism of R regarded as a right R-module. Then $R \approx End(R_R)$ as rings.

Lemma 1.20. An R module $M \neq 0$ is indecomposable iff $End(M)$ has no non-trivial idempotent.

Proposition1.21. Let R be a ring with no nontrivial idempotent. Then R is right almost self- injective if and only if for every $c \in E(R_R)$, either $c \in R$ or there exists $r \in R$ such tha $cr = 1$.

Proof. Assume first R is right almost self –injective module. Lemma 1.19 and 1.20 implies that R_R is an indecomposable module. So R_R is uniform by lemma 1.15.

Let $c \in E(R_R)$ and $I_c: R \to E(R_R)$ be the left multiplication homomorphism. i.e. $I_c(r) = cr$. Then there exits $f : E(R_R) \to E(R_R)$ such that $I_c|_R = f|_R$.

By proposition (1.18) either $f(R) \subseteq R$ or f is an isomorphism and $f^{-1}(R) \subseteq R$. If $f(R) \subseteq R$, then $l_c(1) = c = f(1) \subseteq R$ which implies that $c \in R$. If f is an isomorphism and $f^{-1}(R) \subseteq R$, then there exists $r \in R$ such that $f(r) = 1$. So, $cr = l_c(r) = f(r) = 1$. Conversely, suppose for every $c \in E(R_R)$, either $c \in R$ or there exists$r \in R$ such that $cr = 1$. We claim that $E(R_R)$is uniform.

Let $e \in End(E(R_R)$be an idempotent then for $e(1) \in E(R_R)$, either $e(1) \in R$ or there exists $r \in R$ such that $e(1)r = 1$. If $e(1) \in R$, then $e(1)$ is an idempotent in R and by assumption $e(1) = 0$ or $e(1) = 1$. Hence $e = 0$ or $e = 1_{E(R_R)}$ because $e(r) = e(1.r) = e(1)r = 0.r = 0$ implies $e = 0$or $e(r) = e(1.r) = e(1)r = 1.r = r$

implies $e = 1$. Hence $e = 0$or $= 1_{E(R_R)}$. If $e(1)r = 1$ for some $r \in R$, then $e(r) = 1$. So $e(1) = e\big(e(r)\big) = e^2(r) = e(r) = 1\ implies\ e|_{R_R} = 1_{R_R}$.

We proceed to show that $e = 1_{E(R_R)}$. Suppose that there exists $x \in E(R_R)$ such that $e(x) \neq x$,then $e(x) - x \neq 0$, since $R \subseteq_e E(R_R)$, then there exists $r \in R$, such that $(ex - x)\,r' \neq 0$ and $(ex - x)\,r' \in R$.So $(ex - x)\,r' = e(ex - x)\,r' = (e^2x - ex)\,r' = (ex - ex)\,r = 0$, a contradiction to the fact that $(ex - x)\,r' \neq 0$. Therefore, $e = 1_{E(R_R)}$. This proves E(R_R) is indecomposable and hence uniform. Thus R_R is uniform. Now let $f \in End\,E(R_R)$. Then by assumption$f(1) \in E(R_R)$implies either $f(1) \in R$or $f(1)r = f(r) = 1$.If $f(1) \in R$ implies $f(R) \subseteq R$ because for all $r \in R$, we have $f(r) = f(1.r) = f(1)r \in R$. If $f(r) = 1$ for some r ∈R, then$f|_{rR}$: rR→ R is an isomorphism (because (E(R_R)) is uniform and injective), f is an isomorphism on E(R_R) and f^{-1}(R) =$rR \subseteq R$.By proposition 1.18, R is almost self injective module.

Lemma 1.22. Let M be an indecomposable almost self injective module. Then for every $f, g \in S = End\,(M)$, (i) if $ker\,(f) \subsetneq ker\,(g)$ then $Sg \subsetneq Sf$ (ii) if $ker\,(f) = ker(g)$ then either $Sf \subseteq Sg$ or $Sg \subseteq Sf$.

Proof. Let $\emptyset : f(M) \rightarrow g(M)$ be an R-homorphism defined by $\emptyset(f(m)\,) = g(m)$.

(i) We have $Ker(f) \subsetneq$ ker (g) then $\emptyset$ is not one-one map since there exist $0 \neq m_1 \in Ker\,(g)$ such that $m_1 \notin Ker\,(f)$ such that $\emptyset\big(f(m_1)\big) = g(m_1) = 0$. Since M is almost N-injective module which implies diagram 4 cannot hold and only diagram 3 holds because M is an indecomposable module. By assumption $\emptyset$ can be extended to M. Then there exist $h \epsilon S$ such that $h(f(m)) = \emptyset(f(m))$ for all $m \in M$. Let $I \in S$ be an identity map. Then $I\ o\ g \in Sg$ and $Iog(m) = I(g(m)) = g(m) = \emptyset(f(m)) = h\,(f(m))$ for all $m \in M$ so $h\big(f(m)\big) \in Sf$ implies $Sg \subsetneq Sf$.

Let $ker(f) = ker(g)$. In this case $\emptyset$ is one – one. Because if $\emptyset$ is not one-one implies $\ni m$ such that $f(m) \neq 0$ such that $\emptyset\big(f(m)\big) = 0 = g(m)$which implies $m \in$ ker (g) = ker (f)which is contradiction because $m \notin$ ker (f)). So either $\emptyset$ is extended to an endomorphism $h \epsilon S$ or there exist $\eta \in S$ such that ηoϕ= $I_{f(m)}$. If$\phi = h$ on $f(M)$ then as above $Sg \subseteq Sf$. Ifηoϕ= $I_{f(m)}$. Let I be identity map. That imply $I \in S$. Then $Iof(m) \in Sf$. $Iof(m) = f(m) = \eta o\phi(f(m)) = \eta(\phi(f(m))) = \eta(g(m)) = \eta og(m)$ for all $m \epsilon M$. Thus $Sf \subseteq Sg$.

Lemma 1.23. Let M be an indecomposable almost self-injective module and let $S = End(M)$.Then the left ideal H of S generated by non-isomorphic monomorphisms in S is a two-sided ideal.

Proof. Given that H is left ideal generated by non-isomorphic monomorphism in S. We need only to show that $fg \in H$ for each $g \in S$ and for each non-isomorphism $f \in S$ with $ker\,(f) = 0$. If $ker(fg) \neq 0$ which implies $fg \in H$ (by lemma 1.22).

If $Ker(fg) = 0$ then fg is one-one implies g is one-one. If fg were an isomorphism that implies f would be onto which is contradiction because $f \in H$. That implies $fg \in H$ is non-isomorphic monomorphism.

Theorem 1.24. If M is an indecomposable almost self injective module then $End(M)$ is local.

Proof. Given that M is an indecomposable almost self injective module.To prove $End\ (M)$ is local, we have to prove the set of non-units of $End\ (M)$ forms an ideal.
Let $S = End(M)$ then by lemma1.20, S has only trivial idempotent. Let F be set of all non-isomorphic monomorphism in S. If F is empty, then $\phi \in S$ is an isomorphism iff $ker(\phi) = 0$. Let K be set of non-units in S. We have to prove K is an ideal. Let $h, g \in K$ and suppose that $h + g \in U(S)$ where $U(S)$ is group of units of S. Let $x \in ker\ (h) \cap ker(g)$ then $(h + g)(x) = 0$ implies that $x = 0$. Since M is uniform, either $ker\ (h) = 0$ or $ker\ (g) = 0$. This means either h or g is an isomorphism. Which is contradiction because $h, g \in K$ implies that $h + g \in K$. Let $r \in S$, $h \in K$, if rh is non-units then $rh \in K$. If $rh \in U(S)$ that imply h is one-one . So h is an isomorphism (because F is empty) which is contradiction because $h \in K$. So $rh \in K$. Hence S is local. Suppose F is non-empty. Let $H = \sum_{f \in F} Sf$. By Lemma 1.22, $S \backslash U(S) \subset H$. Now let $h \in H$.We show that h is not invertible. Write $h = \sum_{i=1}^{n} g_i f_i$, where $f_i \in F, g_i \in S$.By Lemma 1.22 $Sf_1, Sf_2, \ldots\ldots\ldots, Sf_n$ are linearly order.So $Sf_1 \subseteq Sf_2 \subseteq \cdots \ldots\ldots \subseteq Sf_n$ after reordering if necessary. Hence $h = g\, f_n$ for some $g \in S$. Now if h is invertible, then f_n is left invertible. Since S has no nontrivial idempotents, f_n is invertible, a contradiction because $f_n \in$F. Thus $H = S \backslash U(S)$. Since H is two- sided ideal of S by lemma 1.23, it follows that S is local.

Theorem 1.25. Let $\{M_i\}_{i=1}^{n}$ be the finite set of indecomposable almost self-injective modules. If M_i is almost M_j-injective for each pair i and j in $\{1,2, \ldots, n\}$ then $\bigoplus_{i=1}^{n} M_i$ is almost self- injective module.

Definition 1.26 .(Generalization *N*-injective modules). In [6], Hanada K. et al. introduced a generalization of relative injectivity.For two modules M and N, M is called generalized N-injective module, if for any submodule X of N and any homomorphism $f: X \to M$, there exist decompositions $N = \overline{N} \oplus \overline{\overline{N}}$, $M = \overline{M} \oplus \overline{\overline{M}}$, a homomorphism $\bar{f}: \overline{N} \to \overline{M}$, and a monomorphism $\bar{f}: \overline{\overline{M}} \to \overline{\overline{N}}$ satisfying properties (*), (**)

(*) X$\subset \overline{N} \oplus g(\overline{\overline{M}})$

(**) For $x \epsilon X$, we express x in $N = \overline{N} \oplus \overline{\overline{N}}$ as $x = \bar{x} \oplus \bar{\bar{x}}$, where $\bar{\bar{x}}$□$\overline{\overline{N}}$, then $f(x) = \bar{f}(\bar{x}) + \bar{\bar{f}}(\bar{\bar{x}})$, where $\bar{\bar{f}} = g^{-1}$.

M is called generalized self-injective module if M is generalized M-injective module.

Proposition 1.27. If M is generalized N- injective module, then M is almost N- injective module.

Proof. Let X be a submodule of N and $f: X \to M$ be a homomorphism. Then there exist decompositions$N = \bar{N} \oplus \bar{\bar{N}}$, $M = \bar{M} \oplus \bar{\bar{M}}$, a homomorphism $\bar{f}: \bar{X} \to \bar{M}$ and a monomorphism $g: \bar{\bar{M}} \to \bar{\bar{N}}$ satisfying the properties $\bar{f}: \bar{X} \to \bar{M}$(*), (**). If f can be extended to N, then $N \neq \bar{N}$ .This means $\bar{\bar{N}} \neq 0$.Define $h : M \to \bar{\bar{N}}$ by $h = g \, o \, \pi_{\bar{\bar{M}}}$ where $\pi_{\bar{\bar{M}}} : M \to \bar{\bar{M}}$ is the canonical projection of M onto $\bar{\bar{M}}$ with respect to the decomposition $M = \bar{M} \oplus \bar{\bar{M}}$. For every $x \in$ X, express x in N = $\bar{N} \oplus \bar{\bar{N}}$ as $x = \bar{x} \oplus \bar{\bar{x}}$, where $\bar{x} \in \bar{N}$ and $\bar{\bar{x}} \in \bar{\bar{N}}$. Then by (**) $hf(x) = h(\bar{f}(\bar{x}) + \overline{\overline{f(\bar{\bar{x}})}})$, where $\bar{\bar{f}} = g^{-1}$

$$= g \, \pi_{\bar{\bar{M}}} \, h(\bar{f}(\bar{x}) + \overline{f(\bar{\bar{x}})})$$
$$= g(\, \overline{\overline{f(\bar{\bar{x}})}}\,)$$
$$= \bar{\bar{x}}$$
$$= \pi_{\bar{\bar{M}}} \text{ o } i_X(\text{x})$$

Hence M is almost N- injective module.

Remark 1.28. Clearly, if M and N are indecomposable modules, then M is almost N-injective module if and only if M is generalized N-injective module.

Definition 1.29. For two modules M and N, M is said to be essentially N-injective module if for every submodule X of N, any homomorphism $f : X \to M$ with $ker\,(f) \subseteq_e X$, then f can be extended to a homomorphism $g : N \to M$ provided $ker\,(f) \subseteq_e N$..

Proposition 1.30. If M is generalized N-injective module, then M is essentially N-injective module.

Proof. Let X be a submodule of N and let $f : X \to M$ be a homomorphism with $ker\, f \subseteq_e X$. Let Y be a submodule of N with $X \oplus Y \subseteq_e N$. Define $g : A = X \oplus Y \to M$ by $g(x + y) = f(x)$. Since $X \oplus Y \subseteq_e N$ and $ker\,(f) \subseteq_e X$, We see ker $(g) \subseteq_e N$. By assumption, there exist decomposition $M = \bar{M} \oplus \bar{\bar{M}}$ and $N = \bar{N} \oplus \bar{\bar{N}}$, a homomorphism $\bar{g} : \bar{N} \to \bar{M}$, and a monomorphism $h: \bar{\bar{M}} \to \bar{\bar{N}}$ satisfying, for $a = \bar{a} + \bar{\bar{a}}$ with $\bar{a} \in \bar{N}$ and $\bar{\bar{a}} \in \bar{\bar{N}}$, $g(a) = (\bar{g}(\bar{a}) + \bar{\bar{g}}(\bar{\bar{a}})$, where $\bar{\bar{g}} = h^{-1}$. Since ker $(g) \subseteq_e N$, we see $Im(h) = 0$ and hence $\bar{\bar{M}} = 0$. Now definef^*: $N = \bar{N} \oplus \bar{\bar{N}} \to M$ by f^* $(\bar{n} + \bar{\bar{n}}) = \bar{g}(\bar{n})$. Then we see $f^*|_X = f$. Thus M is essentially N-injective module.

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THERMAL CONVECTION OF MICROPOLAR FLUID IN THE PRESENCE OF SUSPENDED PARTICLES IN HYDROMAGNETICS IN POROUS MEDIUM

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Abstract

This paper deals with the convection of micropolar fluids heated from below in the presence of suspended particles (fine dust) and uniform vertical magnetic field $\mathbf{H}(0,0,H)$ in a porous medium and using the Boussinesq approximation, the linearized stability theory and normal mode analysis, the exact solutions are obtained for the case of two free boundaries. It is found that the presence of the suspended particles number density, the magnetic field intensity and medium permeability bring oscillatory modes which were non–existent in their absence. It is found that the presence of coupling between thermal, micropolar effects, magnetic field intensity and suspended particles may introduce overstability in the system. Graphs have been plotted by giving numerical values to the parameters accounting for magnetic field intensity $\mathbf{H}(0,0,H)$, the dynamic microrotation viscosity κ and coefficient of angular viscosity γ' to depict the stability characteristics, for both the cases of stationary convection and overstability. It is found that Rayleigh number for the case of overstability and stationary convection increases with increase in magnetic field intensity and decreases with increase in micropolar coefficients and medium permeability, for a fixed wave number, implying thereby the stabilizing effect of magnetic field intensity and destabilizing effect of micropolar coefficients and medium permeability on the thermal convection of micropolar fluids.

Keywords: Micropolar fluid; Magnetic field; Suspended particles (fine dust); Medium permeability; Microrotation; Coefficient of angular velocity.

Introduction

Micropolar theory was introduced by Eringen [5] in order to describe some physical systems which do not sastisfy the Navier Stokes equations. These fluids are able to describe the behaviour of colloidal solutions, liquid crystals; animal blood etc.The equations governing the flow of micropolar fluid theory involve a spin vector and a microinertia tensor in addition to velocity vector. A generalization of the theory including thermal effects has been developed by Kazakia and Ariman [7] and Eringen [6]. Micropolar fluid stabilities have become an important field of research these days. A particular stability problem is the Rayleigh-Bénard instability in a horizontal thin layer of fluid heated from below. A detailed account of thermal convection in a horizontal thin layer of Newtonian fluid heated from below has been given by Chandrasekhar [4]. Ahmadi [1] and Pérez-Garcia et al. [13] have studied the effects of the microstructures on the thermal convection and have found that in the absence of coupling between thermal and micropolar effects, the principle of exchange of stabilities may not be fulfilled and consequently micropolar fluids introduce oscillatory motions. The existence of oscillatory motions in micropolar fluids has been depicted by Lekkerkerker in liquid crystals [9, 10], Bradley in dielectric fluids [3] and Laidlaw in binary mixture [11]. In the study of problems of thermal convection, it is frequent practice to simplify the basic equations by introducing an approximation which is attributed to Boussinesq [2]. In geophysical situations, the fluid is often not pure but contains suspended particles. Saffman [17] has considered the stability of laminar flow of a dusty gas. Scanlon and Segel [18] have considered the effects of suspended particles on the onset of Bénard convection, whereas Sharma et al.[19] have studied the effect of suspended particles on the onset of Bénard convection in hydromagnetics and found that the critical Rayleigh number was reduced because of the heat capacity of the particles. The separate effects of suspended particles, rotation and solute gradient on thermal instability of fluids saturating a porous medium have been discussed by Sharma and Sharma [20]. The suspended particles were thus found to destabilize the layer. Palaniswami and Purushotham [14] have studied the stability of shear flow of stratified fluids with the fine dust and found that the presence of dust particles increases the region of instability. On the other hand, multiphase fluid systems are concerned with the motion of liquid or gas containing immiscible inert identical particles.

The theoretical and experimental results of the onset of themal instability (Bénard convection) in a fluid layer under varying assumptions of hydromagnetics, has been depicted in a treatise by Chandrasekhar [4]. Lapwood [8] has studied the convective flow in porous medium using linearized stability theory. The Rayleigh instability in flow through a porous medium has been considered by Wooding [15]. The problem of thermal convection in a fluid in porous medium is of importance in geophysics, soil–science, ground–water, hydrology and astrophysics. The physical property of comets, meteororites and inter–planetary dust strongly suggests the importance of porosity in the astrophysical context (McDonnel [12]). Moreover, Saffman and Taylor [16] have shown that the motion in a Hele–Shaw cell is mathematically analogous to two dimensional flow in porous medium. In recent years, there has been a considerable interest in the study of breakdown of the stability of a layer of a fluid subjected to a vertical temperature gradient in a porous medium and also in the possibility of convective flow.

When a fluid permeates a porous material, the gross effect is represented by Darcy's law. As a result of this macroscopic law, the usual viscous term in the equations of motion of microscopic fluid is replaced by the resistance term $\left[-\frac{1}{k_1}(\mu+\kappa)\,\mathbf{q}\right]$, where μ and κ are viscosity and dynamic microrotation viscosity respectively, k_1 is the medium permeability and $\mathbf{q}$ is the Darcian (filter) velocity of the fluid. Sharma and Gupta [21] have studied the thermal convection in micropolar fluids in porous medium and have found that medium permeability has stabilizing effect on stationary convection and destabilizing effect on the overstable case. Sharma and Kumar [22] have studied the thermal instability of micropolar fluids in hydromagnetics in porous medium. Keeping in mind the importance and relevance of porosity and hydromagnetics in chemical engineering, geophysics and biomechanics, thermal instability of micropolar fluids in the presence of a uniform vertical magnetic field to include the effect of suspended particles (dust particles) in porous medium has been considered in the present paper.

Mathematical formulation and analysis

Consider an infinite, horizontal layer of an incompressible electrically conducting micropolar fluid of thickness d permeated with suspended particles (or fine dust) in an isotropic and homogeneous medium of porosity ε and medium permeability k_1. This fluid-particles layer is heated from below but convection sets in when the temperature gradient $\left(\beta=\left|{}^{dT}\!/_{dz}\right|\right)$ between the lower and upper boundaries exceeds a certain critical value. A uniform vertical magnetic field $\mathbf{H}(0,0,H)$ pervades the system. This is the Rayleigh-Bénard instability problem in micropolar fluids. Both the boundaries are taken to be free and perfect conductor of heat. The mass, momentum, internal angular momentum, internal energy balance equations using the Boussinesq approximation are

$$\nabla\cdot\mathbf{q}=0, \tag{1}$$

$$\frac{1}{\varepsilon}\left(\frac{\partial}{\partial t}+\frac{\vec{q}}{\varepsilon}\cdot\nabla\right)\mathbf{q}=-\frac{1}{\rho_0}\nabla p-\frac{1}{\rho_0\,k_1}(\mu+\kappa)\mathbf{q}+\frac{\kappa}{\rho_0}\nabla\times\vartheta-\left(1+\frac{\delta\rho}{\rho_0}\right)g\,\hat{e}_z+\frac{1}{\rho_0}\frac{KN}{\varepsilon}(\mathbf{u}-\mathbf{q})$$
$$+\frac{1}{\rho_0}\frac{\mu_e}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H}, \tag{2}$$

$$\rho_0 j_1\left(\frac{\partial}{\partial t}+\frac{\mathbf{q}}{\varepsilon}\cdot\nabla\right)\vartheta=(\varepsilon'+\beta')\nabla(\nabla\cdot\vartheta)+\gamma'\nabla^2\vartheta+\frac{\kappa}{\varepsilon}\nabla\times\mathbf{q}-2\kappa\vartheta, \tag{3}$$

$$[\rho_0 c_v\varepsilon+\rho_s c_s(1-\varepsilon)]\frac{\partial T}{\partial t}+\rho_0\, c_v\,\mathbf{q}\cdot\nabla T=k_T\nabla^2 T+\delta(\nabla\times\vartheta)\cdot\nabla T. \qquad 4)$$

where $\mathbf{q}$, ϑ, p, ρ, $\mathbf{g}$, ρ_0, μ_e and $\mathbf{u}$, denote the filter (seepage) velocity, the spin, the pressure, the fluid density, the acceleration due to gravity, the reference density, magnetic permeability and velocity of the suspended particles, respectively. $N(\mathbf{x},t)$ denotes the number density of dust particles and κ is the dynamic microrotation viscosity. $\mathbf{x}=(x,y,z)$. $K=6\pi\mu r$, r being the particle radius, is the Stokes drag coefficient and k_T, c_v, c_s c_{pt}, δ, j_1 denote, respectively, the thermal conductivity, the specific heat at constant volume, the heat capacity of solid matrix , the heat capacity of particles, the coefficient giving account of coupling between spin and heat flux , and microinertial constant. ε', β', γ' are the coefficients of angular viscosity.

Assuming dust particles of uniform size, spherical shape and small relative velocities between the two phases (fluid and particles), the net effect of the particles on the fluid is equivalent to an extra body force term per unit volume $KN(\mathbf{u}-\mathbf{v})$, as has been taken in equation [2]. We also use the Boussinesq approximation by allowing the density to change only in the gravitational body force term.

The density equation of the state is

$$\rho=\rho_0[1-\alpha(T-T_0)] \quad ,$$

where ρ_0, T_0 are reference density, reference temperature at the lower boundary and α is the coefficient of thermal expansion.

Since the force exerted by the fluid on the particles is equal and opposite to that exerted by the particles on the fluid. The distance between the particles is assumed to be so large compared with their diameter that interparticle reactions are ignored. The buoyancy force on the particles is also neglected. If mN is the mass of suspended particles per unit volume, then the equations of motion and continuity for the particles, under the above assumptions, are

$$mN\left(\frac{\partial}{\partial t}+\frac{\mathbf{u}}{\varepsilon}\cdot\nabla\right)\mathbf{u}=KN(\mathbf{q}-\mathbf{u}), \tag{5}$$

$$\varepsilon\frac{\partial N}{\partial t}+\nabla\cdot(N\mathbf{u})=0 \tag{6}$$

The Maxwell's equations yield

$$\varepsilon \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{q} \times \mathbf{H}) + \varepsilon\, \eta\, \nabla^2 \mathbf{H}, \tag{7}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{8}$$

where $\eta \left(= \dfrac{1}{4\pi\mu_e \sigma} \right)$ is called resistivity and σ is electrical conductivity.

In the quiescent state, the solution of equations [1] – [8] is

$$\mathbf{q} = 0,\ \mathbf{u} = 0,\ \vartheta = 0,\ N = N_0 \text{(constant)},\ T = T_0 - \beta z,\ \rho = \rho_0 (1 + \alpha\, \beta\, z),$$

$$p = p_0 - g\, \rho_0 \left(z + \frac{\alpha\, \beta\, z^2}{2} \right), \tag{9}$$

where p_0 is the pressure at $z = 0$ and $\beta = \dfrac{T_0 - T_1}{d}$ $(T_0 > T_1)$ is the magnitude of uniform

temperature gradient.

Assume small perturbations around the basic state, and let $\mathbf{q} = (u, v, w)$, $\mathbf{u} = (\ell, r, s)$, $\boldsymbol{\omega}$, p', ρ', θ and $\mathbf{h}\left(h_x, h_y, h_z \right)$ denote, respectively, the perturbations on fluid velocity $\mathbf{q}$, particles velocity $\mathbf{u}$, spin ϑ, pressure p, density ρ, temperature T and magnetic field $\mathbf{H}(0, 0, H)$, so that the change in density ρ' caused by the perturbations θ in temperature is given by

$$\rho' = -\rho_0 \alpha\, \theta. \tag{10}$$

Then the linearized perturbation equations of the microplar fluid become

$$\nabla \cdot \mathbf{q} = 0, \tag{11}$$

$$\frac{\rho_0}{\varepsilon} \left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon} \cdot \nabla \right) \mathbf{q} = -\nabla p' - \frac{1}{k_1} (\mu + \kappa) \mathbf{q} + \kappa (\nabla \times \boldsymbol{\omega}) + g\, \rho_0 \alpha\, \theta\, \hat{e}_z + \frac{K N_0}{\varepsilon} (\mathbf{u} - \mathbf{q}) +$$

$$\frac{\mu_e}{4\pi} (\nabla \times \mathbf{h}) \times \mathbf{H}, \tag{12}$$

$$\rho_0 j_1 \left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon} \cdot \nabla \right) \boldsymbol{\omega} = (\varepsilon' + \beta') \nabla (\nabla \cdot \boldsymbol{\omega}) + \gamma' \nabla^2 \boldsymbol{\omega} + \frac{\kappa}{\varepsilon} \nabla \times \mathbf{q} - 2\, \kappa \boldsymbol{\omega}, \tag{13}$$

$$\left[\rho_0\, c_v \varepsilon + \rho_s c_s (1-\varepsilon)\right] \left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon}\cdot\nabla\right)\theta = \beta\,(w + h_1 s) + k_T \nabla^2\theta + \delta\left[\nabla\theta\cdot(\nabla\times\boldsymbol{\omega}) - (\nabla\times\boldsymbol{\omega})_z\cdot\beta\right],$$

(14)

$$\varepsilon\frac{\partial \mathbf{h}}{\partial t} = \nabla\times(\mathbf{q}\times\mathbf{h}) + \varepsilon\,\eta\nabla^2\mathbf{h}, \tag{15}$$

$$\nabla\cdot\mathbf{h} = 0, \tag{16}$$

$$mN_0\left(\frac{\partial}{\partial t} + \frac{\mathbf{u}}{\varepsilon}\cdot\nabla\right)\mathbf{u} = KN_0(\mathbf{q}-\mathbf{u}), \tag{17}$$

$$\varepsilon\frac{\partial M}{\partial t} + \nabla\cdot\mathbf{u} = 0, \tag{18}$$

where $H_1 = 1 + h_1,\, h_1 = \dfrac{fc_{pt}}{c_v},\quad f = \dfrac{mN_0}{\rho_0}$ and $M = \dfrac{N}{N_0}$.

Using the non–dimensional numbers

$$z = z^* d,\quad \theta = \beta\, d\theta^*,\quad t = \frac{\rho_0 d^2}{\mu}t^*,\quad \mathbf{q} = \frac{\kappa_T}{d}\mathbf{q}^*,\ \nabla = \frac{\nabla^*}{d}$$

$$\mathbf{u} = \frac{\kappa_T}{d}\mathbf{u}^*,\quad p = \frac{\mu\kappa_T}{d^2}p^*,\quad \boldsymbol{\omega} = \frac{\kappa_T}{d^2}\boldsymbol{\omega}^*,\ \ \mathbf{h} = \left(\frac{\mu\kappa_T}{d^2}\right)^{\frac{1}{2}}\mathbf{h}^* \tag{19}$$

Equations [11] - [18] in the non-dimensional form are

$$\nabla\cdot\mathbf{q} = 0, \tag{20}$$

$$\frac{1}{\varepsilon}\left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon}\cdot\nabla\right)\mathbf{q} = -\nabla p' - \frac{1}{\bar{k}_1}(1+K_1)\nabla^2\mathbf{q} + K_1\nabla\times\boldsymbol{\omega} + \hat{e}_z\, R\theta + \frac{N_2}{\varepsilon}(\mathbf{u}-\mathbf{q}) +$$

$$\frac{\mu_e}{4\pi}(\nabla\times\mathbf{h})\times\mathbf{H}, \tag{21}$$

$$\bar{j}_2\left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon}\cdot\nabla\right)\boldsymbol{\omega} = C_1'\,\nabla(\nabla\cdot\boldsymbol{\omega}) - C_0'\nabla\times(\nabla\times\boldsymbol{\omega}) + K_1\left(\frac{1}{\varepsilon}\nabla\times\mathbf{q} - 2\boldsymbol{\omega}\right), \tag{22}$$

$$EH_1 p_1\left(\frac{\partial}{\partial t} + \frac{\mathbf{q}}{\varepsilon}\cdot\nabla\right)\theta = \beta\,(w + h_1 s) + \kappa_T\nabla^2\theta + \bar{\delta}\left[\nabla\theta\cdot(\nabla\times\boldsymbol{\omega}) - (\nabla\times\boldsymbol{\omega})_z\right], \tag{23}$$

$$\varepsilon\frac{\partial \mathbf{h}}{\partial t} = \nabla\times(\mathbf{q}\times\mathbf{h}) + \frac{\varepsilon}{p_2}\nabla^2\mathbf{h}, \tag{24}$$

$$\nabla \cdot \mathbf{h} = 0, \tag{25}$$

$$\left[a\left(\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla \right) + 1 \right] \mathbf{u} = \mathbf{q}, \tag{26}$$

where the following non-dimensional parameters are introduced

$$K_1 = \frac{\kappa}{\mu}, \quad \bar{j}_2 = \frac{j}{d^2}, \quad \bar{\delta} = \frac{\delta}{\rho_0 c_v d^2}, \quad C_0' = \frac{\gamma'}{\mu d^2}, \quad C_1' = \frac{\varepsilon' + \beta' + \gamma'}{\mu d^2},$$

$$E = \varepsilon + (1-\varepsilon)\frac{\rho_s c_s}{\rho_0 c_v}, \quad \bar{k}_1 = \frac{k_1}{d^2}, \quad N_2 = KN_0 \frac{d^2}{\mu}, \quad a = \frac{m}{Kd^2} \quad \frac{\mu}{\rho_0},$$

$$R = \frac{g\,\alpha\,\beta\,\rho_0\,d^4}{\mu\,\kappa_T}, p_1 = \frac{\upsilon}{\kappa_T}, p_2 = \frac{\upsilon}{\eta}, \quad \kappa_T = \frac{k_T}{\rho_0 c_v}. \tag{27}$$

Eliminating s between equations [23] and [26] and applying the curl operator twice to resulting equation, we obtain

$$L_2\left[EH_1 p_1 \frac{\partial}{\partial t} - \nabla^2 \right]\theta = \left(a\frac{\partial}{\partial t} + H_1 \right)\beta\, w - L_2 \bar{\delta}\Omega_z. \tag{28}$$

Eliminating $\mathbf{u}$ between equations [21] and [26] and on linearizing, we obtain

$$\varepsilon^{-1} L_1\, \mathbf{q} = L_2\left[-\nabla p' - \frac{1}{\bar{k}_1}(1+K_1)\mathbf{q} + K_1 \nabla \times \boldsymbol{\omega} + R\theta\,\hat{e}_z + \frac{\mu_e}{4\pi}(\nabla \times \mathbf{h}) \times \mathbf{H} \right] \quad ,(29)$$

where

$$L_1 = a\frac{\partial^2}{\partial t^2} + F\frac{\partial}{\partial t}, \quad L_2 = a\frac{\partial}{\partial t} + 1 \qquad \text{and}$$

$$F = f + 1.$$

Applying the curl operator to equations [21], [22] and [24] taking z–component, we get

$$\varepsilon^{-1} L_2 \frac{\partial}{\partial t}\zeta_z + \varepsilon^{-1} N_2 \zeta_z (L_2 - 1) = -\frac{1}{\bar{k}_1}(1+K_1)\nabla^2 \zeta_z\, L_2 + \frac{\mu_e H}{4\pi}\frac{\partial \xi_z}{\partial z} L_2, \tag{30}$$

$$\bar{j}_2 \frac{\partial \Omega_z}{\partial t} = C_0' \nabla^2 \Omega_z - K_1\left(\frac{1}{\varepsilon}\nabla^2 w + 2\Omega_z \right), \tag{31}$$

$$\varepsilon \frac{\partial \xi_z}{\partial t} = H \frac{\partial}{\partial t} \zeta_z + \frac{\varepsilon}{p_2} \nabla^2 \xi_z, \tag{32}$$

where $\xi_z = (\nabla \times \mathbf{h})_z$, $\zeta_z = (\nabla \times \mathbf{q})_z$ are the $z-$components of current density and vorticity, respectively. K_1 and C_0' account for coupling between vorticity and spin effects and spin diffusion, respectively.

Taking the z –component of equations [24], we get

$$\varepsilon \frac{\partial h_z}{\partial t} = H \frac{\partial w}{\partial z} + \frac{\varepsilon}{p_2} \nabla^2 h_z. \tag{33}$$

Applying the curl operator twice to equations [21] and taking z –component, we get

$$\varepsilon^{-1} L_1 \nabla^2 w = L_2 \left[R \nabla_1^2 \theta - \frac{1}{\bar{k}_1} (1 + K_1) \nabla^2 w + K_1 \nabla^2 \Omega_z + \frac{\mu_e H}{4\pi} \frac{\partial}{\partial z} \nabla^2 h_z \right], \tag{34}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Omega_z = (\nabla \times \boldsymbol{\omega})_z$. (35)

The boundaries are considered to be free. The case of two free boundaries is little artificial except in astrophysical situations but it enables us to find analytical solutions.

Thus the boundary conditions appropriate to problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial}{\partial z} (\nabla \times \mathbf{q})_z = 0, (\nabla \times \mathbf{h})_z = (\nabla \times \boldsymbol{\omega})_z = 0, \ \theta = \frac{\partial h_z}{\partial z} = 0 \text{ at } z = 0 \text{ and } z = d. \tag{36}$$

Now we analyze the perturbations into a complete set of normal modes and then examine the stability of each of these modes individually. For the system of equations [28], [30]-[34], the analysis can be made in terms of two dimensional periodic waves of assigned wave numbers. Thus we ascribe to all quantities describing the perturbation dependence on x, y and t of the form $\exp\left[i (k_x x + k_y y) + nt \right]$, where k_x, k_y are the wave numbers along the x- and y- directions, respectively, $k = \left(k_x^2 + k_x^2 \right)^{1/2}$ is the resultant wave number, n is the stability parameter which can be, complex, in general. The solution of the stability problem requires the specifications of the state for each k. The above considerations allow us to suppose that the perturbation quantities have the form

$$[w, \Omega_z, \zeta_z, \xi_z, \theta, h_z] = [W(z), \Omega_2(z), Z(z), G(z), \Theta(z), B(z)] \exp\left(ik_x x + ik_y y + nt \right), \tag{37}$$

then the equations [28], [30]-[34], become

$$\left(an+1\right)\left\{EH_1 p_1 n-\left(D^2-k^2\right)\right\}\Theta=\left(an+H_1\right)W-\left(an+1\right)\bar{\delta}\,\Omega_2\,, \tag{38}$$

$$\left(D^2-k^2\right)\left\{\left(an^2+Fn\right)+\frac{1}{\bar{k}_1}\left(an+1\right)\left(1+K_1\right)\left(D^2-k^2\right)\right\}W=\left(an+1\right)\left\{-Rk^2\Theta+K_1\left(D^2-k^2\right)\Omega_2\right.$$

$$\left.+\frac{\mu_e H}{4\pi}\left(D^2-k^2\right)DB\right\}, \tag{39}$$

$$\left\{\varepsilon^{-1}\left(an^2+Fn\right)+\left(an+1\right)\frac{1}{\bar{k}_1}\left(1+K_1\right)\right\}Z=\frac{\mu_e H}{4\pi}\left(an+1\right)DG\,, \tag{40}$$

$$\left\{\ell_1 n+2A-\left(D^2-k^2\right)\right\}\Omega_2=-A\varepsilon^{-1}\left(D^2-k^2\right)W, \tag{41}$$

$$\left\{n-\frac{1}{p_2}\left(D^2-k^2\right)\right\}G=\varepsilon^{-1}H\;DZ\,, \tag{42}$$

$$\left\{n-\frac{1}{p_2}\left(D^2-k^2\right)\right\}B=\varepsilon^{-1}H\;DW\,, \tag{43}$$

where $A=\dfrac{K_1}{C_0'},\ \ell_1=\bar{j}_2\dfrac{A}{K_1}.$

The boundary conditions [36] transform to

$$W=0,\ D^2W=0,\ DZ=0,\ G=0,\ \Omega_2=0,\ \Theta=0,\ DB=0\ \text{ at }\ z=0 \text{ and } 1. \tag{44}$$

Using boundary conditions (44), equations (38)–(43) transform to

$$D^2\Theta=0,\ D^2\Omega_2=0,\ D^3Z=0,\ D^3G=0,\ D^3B=0\,, \tag{45}$$

Differentiating [39] twice with respect to z and using boundary conditions [45], it can be shown that $D^4W=0$. It can be shown from equations.[38]–[43] and boundary conditions [44], [45] that all even order derivatives of W vanish on the boundaries .The proper solution of W belonging to the lowest mode is

$$W=W_0\sin\pi z\,, \tag{46}$$

where W_0 is a constant.

Eliminating Θ, Γ, Ω_2 from equations [38] - [43] and substituting the solution given by equation [46], we obtain the dispersion relation

$$Rk^2\left\{n+\frac{b}{p_2}\right\}\left\{(an+H_1)(\ell_1 n+2A+b)-(an+1)\varepsilon^{-1}\bar{\delta}Ab\right\}=b\left\{\varepsilon^{-1}\left(an^2+Fn\right)+\frac{1}{\bar{k}_1}(an+1)(1+K_1)\right\}$$

$$\left(EH_1p_1n+b\right)(\ell_1 n+2A+b)\left\{n+\frac{b}{p_2}\right\}-\varepsilon^{-1}K_1Ab^2(an+1)(EH_1p_1n+b)\left\{n+\frac{b}{p_2}\right\}+$$

$$\frac{H^2\pi}{4}\left(EH_1p_1nb+b^2\right)(an+1)\varepsilon^{-1}\left(\ell_1 n+2A+b\right). \qquad (47)$$

where $b=\pi^2+k^2$.

The case of oscillatory modes

Here we examine the possibility of oscillatory modes, if any, in the stability problem due to the presence of magnetic field intensity and suspended particles number density. Equating the imaginary parts of equation [47], we have

$$n_i\Bigg[n_i^4 EabH_1p_1\ell_1\varepsilon^{-1}+n_i^2\Bigg(-2Aab^2\varepsilon^{-1}-ab^3\varepsilon^{-1}-2EH_1p_1A\frac{ab^2}{p_2}\varepsilon^{-1}-2EH_1p_1A\frac{ab^3}{p_2}\varepsilon^{-1}-\frac{a^2b^3}{p_2}\ell_1\varepsilon^{-1}$$

$$-EFH_1p_1b\,\varepsilon^{-1}-EFH_1p_1b^2\varepsilon^{-1}-F\,b^2\ell_1\varepsilon^{-1}-EH_1p_1\ell_1\varepsilon^{-1}\frac{b^2}{p_2}+\frac{1}{\bar{k}_1}\Bigg\{-2AEH_1p_1ab-EH_1p_1ab^2-\frac{ab^2\ell_1}{p_2}$$

$$-EH_1p_1\ell_1\frac{ab^2}{p_2}-2AEH_1p_1abK_1-EH_1p_1K_1b^2-ab^2\ell_1K_1-EH_1p_1\ell_1bK_1+\frac{b^3a}{p_2}K_1\Bigg\}+Rk^2a\ell_1+$$

$$2A\frac{b^5}{p_2}F\varepsilon^{-1}+\frac{b^4}{p_2}F\varepsilon^{-1}\Bigg)+\frac{ab^4}{p_2}+\frac{2b^3}{p_2}\left(Aa+\frac{1}{\bar{k}_1}\left\{AaK_1+AEH_1p_1K_1\right\}\right)+\frac{b^2}{p_2}\Bigg(\frac{1}{\bar{k}_1}2Ab^3EH_1p_1K_1$$

$$+Rk^2\varepsilon^{-1}a\,\bar{\delta}A-Rk^2a\Bigg)+b^2\frac{1}{\bar{k}_1}2AK_1+b\left(Rk^2\left\{-H_1+\bar{\delta}A\,\varepsilon^{-1}-\frac{2A}{p_2}a-\frac{1}{p_2}H_1\,\ell_1\right\}\right)-2Rk^2AH_1\Bigg]=0.$$

$$(48)$$

It is evident from equation [48] that n_i may be either zero or non-zero, meaning thereby that the modes may be non-oscillatory or oscillatory. In the absence of suspended particles and solute parameter equation [48] reduces to

$$n_i\left(2A\,b^2K_1+R\,k^2\bar{\delta}A\,b\right)=0 \qquad (49)$$

and term within the brackets is definitely positive, which implies that $n_i = 0$. Therefore, the oscillatory modes are not allowed and principal of exchange of stabilities is satisfied for porous medium in the absence of suspended particles and magnetic field. The presence of the suspended particles number density, the magnetic field intensity and medium permeability bring oscillatory modes (as n_i may not be zero) which were non–existent in their absence.

The case of overstability

The present section is devoted to the possibility that instability may occur as overstability. Since we wish to determine the Rayleigh number for onset of instability via a state of pure oscillations, it suffices to find the conditions for which equation [47] will admit of solutions with n_i real. Substituting $n = in_i$ in equation [47], and then equating the real and imaginary parts of equation [47] we obtain

$$Rk^2\left[\frac{b^2}{p_2}\left\{2H_1A + b\left(1-\varepsilon^{-1}\bar{\delta}A\right)\right\} - n_i^2\left\{b\ell_1\left(1+\frac{a}{p_2}\right) + a\left\{2AH_1E + b\left(1-\varepsilon^{-1}\bar{\delta}A\right)\right\}\right\}\right] =$$

$$n_i^4\left[EH_1p_1\ell_1b^2\varepsilon^{-1}a\frac{(1+K_1)}{\bar{k}_1} + \varepsilon^{-1}ab\left\{b\ell_1\left(1+\frac{EH_1p_1}{p_2}\right)\right\} + EH_1p_1\varepsilon^{-1}\left(2aA + F\ell_1\right)\right]$$

$$-n_i^2\left[\left\{(2A+b)\varepsilon^{-1}\left(1+\frac{EH_1p_1}{p_2}\right) + \frac{EH_1p_1\ell_1}{p_2}\right\}\left\{2aAEH_1\frac{(1+K_1)}{\bar{k}_1}b^3 + b^3\varepsilon^{-1}\left(2aA + F\ell_1\right)\right\}\right.$$

$$\left\{EH_1p_1\frac{(1+K_1)}{\bar{k}_1}\varepsilon^{-1} + \frac{\varepsilon^{-1}aE}{p_2}\right\} + b^4a\ell_1E\frac{(1+K_1)}{\bar{k}_1}\left(1+\frac{EH_1p_1}{p_2}\right) - K_1A\varepsilon^{-1}b^3$$

$$\left.\left\{Ep_1H_1 + a\varepsilon^{-1}\left(1+\frac{Ep_1H_1}{p_2}\right)\right\}\right] + \frac{H^2\pi}{4}\varepsilon^{-1}\left[-n_i^2\left\{b\ell_1\left(EH_1p_1 + a\right) + EH_1p_1\varepsilon^{-1}(2A+b)\right\}\right]$$

$$+\left[\frac{1}{p_2}\left\{\frac{(1+K_1)}{\bar{k}_1} - \varepsilon^{-1}K_1A\right\}b^5 + \frac{b^4}{p_2}\left\{2A\frac{(1+K_1)}{\bar{k}_1}\right\}\right] \tag{50}$$

and

$$Rk^2\left[-a\ell_1n_i^3 + 2AH_1n_i + H_1bn_i - n_i\bar{\delta}Ab\varepsilon^{-1} + \frac{2Ab}{p_2}an_i + \frac{b^2}{p_2}an_i + \frac{b}{p_2}H_1\ell_1n_i - \frac{b^2}{p_2}\varepsilon^{-1}an_i\bar{\delta}A\right] =$$

$$EabH_1p_1\ell_1n_i^5\varepsilon^{-1} - 2Aab^2n_i^3\varepsilon^{-1} - ab^3n_i^3\varepsilon^{-1} - 2EH_1p_1n_i^3A\frac{ab^2}{p_2}\varepsilon^{-1} - 2EH_1p_1n_i^3A\frac{ab^3}{p_2}\varepsilon^{-1} - \frac{a^2b^3}{p_2}$$

$$\ell_1 n_i^3 \varepsilon^{-1} - EFH_1 p_1 n_i^3 b\varepsilon^{-1} - EFn_i^3 H_1 p_1 b^2 \varepsilon^{-1} - Fb^2 \ell_1 n_i^3 \varepsilon^{-1} - EH_1 p_1 \ell_1 n_i^3 \varepsilon^{-1} \frac{b^2}{p_2} + 2A\frac{b^5}{p_2}$$

$$Fn_i \varepsilon^{-1} + \frac{b^4}{p_2} Fn_i \varepsilon^{-1} + \frac{1}{\bar{k}_1}\Bigg[-2AEH_1 p_1 abn_i^3 - EH_1 p_1 n_i^3 ab^2 - \frac{an_i^3 b^2 \ell_1}{p_2} - EH_1 p_1 \ell_1 n_i^3 \frac{ab^2}{p_2}$$

$$+ 2Aan_i \frac{b^3}{p_2} + \frac{ab^4}{p_2} n_i K_1 - 2AEH_1 p_1 n_i^3 abK_1 - EH_1 p_1 n_i^3 K_1 b^2 - ab^2 \ell_1 n_i^3 K_1 + 2Aan_i \frac{b^3}{p_2} K_1$$

$$+ \frac{ab^4}{p_2} n_i K_1 - EH_1 p_1 n_i^3 \ell_1 b + 2Ab^2 n_i + b^3 n_i + 2AEH_1 p_1 n_i \frac{b^2}{p_2} + EH_1 p_1 n_i \frac{b^3}{p_2} + \frac{b^3}{p_2} \ell_1 n_i K_1$$

$$- EH_1 p_1 n_i^3 \ell_1 bK_1 + 2Ab^2 n_i K_1 + 2AEH_1 p_1 n_i \frac{b^2}{p_2} K_1 + 2AEH_1 p_1 n_i \frac{b^3}{p_2} K_1 + \frac{b^3}{p_2} n_i K_1 \Bigg]. \tag{51}$$

Eliminating R between equation (50) and (51), we get

$$n_i^6 \left[b^2 \left\{ \varepsilon^{-1} a^2 H_1^2 \ell_1^2 \frac{(1+K_1)}{\bar{k}_1} \right\} + b \left\{ \frac{\varepsilon^{-1} aEH_1 p_1}{\bar{k}_1} \left(EH_1 \ell_1 - ab\varepsilon^{-1} \bar{\delta} A - F\ell_1 \right) - \varepsilon^{-1} b^3 a\ell_1 H_1 E \left(\frac{E}{p_2} + \frac{(1+K_1)}{\bar{k}_1} \right) \right\} \right]$$

$$+ n_i^4 \Bigg[b^5 \left\{ EH_1 p_1 a^2 \left(1 - \varepsilon^{-1} \bar{\delta} A\right) + H_1 p_1 \ell_1 \bar{\delta} A \frac{E}{p_2} \frac{1+K_1}{\bar{k}_1} + \frac{Ep_1}{p_2} \bar{\delta} A (H_1 - 1) \right\} + b^4 \Bigg\{ 2EH_1 p_1 a^2 A \frac{(1+K_1)}{\bar{k}_1}$$

$$- \frac{Ep_1}{p_2} a^2 p_1 H_1^2 (H_1 - 1) \Bigg\} + b^3 \Bigg\{ \frac{1}{p_2^2} EH_1 p_1 aF\left(1 - \varepsilon^{-1} \bar{\delta} A\right) + EH_1 p_1 \ell_1^2 a^2 \frac{1}{p_2^2} (H_1 - 1) - EH_1 p_1 \ell_1 K_1 a$$

$$\left(2 - \varepsilon^{-1} \bar{\delta} A\right) + \frac{E}{p_2^2} H_1 p_1 a \ell_1^2 \left(1 - \varepsilon^{-1} \bar{\delta} A\right) \Bigg\} + b^2 \Bigg\{ \frac{-aE^2}{p_2} 2AF\ell_1 (H_1 - 1) - \frac{H^2 \pi}{4} \varepsilon^{-1} \ell_1 a \left(a\ell_1 - \frac{Ep_1}{p_2} a\ell_1 + p_1 \varepsilon^{-1} \bar{\delta} A \right)$$

$$- 2Aa\ell_1 H_1 E^2 \frac{(1+K_1)}{\bar{k}_1} + 2AE^2 \varepsilon^{-1} (2 - K_1 Ep_1) \Bigg\} + b \Bigg\{ \frac{1}{p_2^2} - 2A^2 E^2 aH_1 p_1 (H_1 - F) - \frac{2H^2 \pi}{4} \varepsilon^{-1} a\ell_1 A$$

$$\left(a\ell_1 - \frac{Ep_1}{p_2} + p_1 \varepsilon^{-1} \bar{\delta} A \right) \Bigg\} \Bigg] + n_i^2 \Bigg[b^7 \left\{ -E^2 H_1 p_1 \frac{a}{p_2^2} \left(1 - \varepsilon^{-1} \bar{\delta} A\right) - \varepsilon^{-1} \bar{\delta} A \frac{aE}{p_2^2} \frac{(1+K_1)}{\bar{k}_1} \right\} + b^6 \Bigg\{ \frac{E^2 a^2}{p_2^2} \varepsilon^{-1} \bar{\delta} A$$

$$\frac{(1+K_1)}{\bar{k}_1} - EH_1 p_1 \ell_1 \frac{1}{p_2^2} \left(2 - \varepsilon^{-1} \bar{\delta} A\right) - \frac{2Ep_1}{p_2} H_1^2 aA \frac{(1+K_1)}{\bar{k}_1} - \frac{H^2 \pi}{4} E^2 \frac{H_1 p_1}{p_2} \left(2 - \varepsilon^{-1} \bar{\delta} A\right) - EH_1 p_1 \bar{\delta} A$$

$$\frac{1}{p_2^2} \varepsilon^{-1} (F - aK_1) - \bar{\delta} A \varepsilon^{-1} H_1 \frac{1}{p_2} \ell_1 \frac{(1+K_1)}{\bar{k}_1} \Bigg\} + b^5 \Bigg\{ - \frac{E^2 H_1 p_1}{p_2^2} a\ell_1 (H_1 - 1) + \varepsilon^{-1} H_1 p_1 \frac{a^2}{p_2^2} E^2 \frac{(1+K_1)}{\bar{k}_1}$$

$$+\bar{\delta}A\frac{\varepsilon^{-1}a}{p_2}\frac{(1+K_1)}{\bar{k}_1}-\frac{E^2\ell_1^2}{p_2^2}(2aA+F\ell_1)-2A^2\varepsilon^{-1}\frac{\bar{\delta}E}{p_2}(H_1-F)+E^2H_1ap_1\frac{1}{\bar{k}_1p_2^2}(F-aK_1)+\frac{\varepsilon^{-1}K_1aA}{p_2^2}$$

$$(E\ell_1+H_1p_1\varepsilon^{-1}\bar{\delta}A)+\frac{4AaE^2}{p_2^2}\left(1+H_1p_1E\frac{(1+K_1)}{\bar{k}_1}-EK_1p_1\right)+2\left\{1+EH_1p_1\frac{(1+K_1)}{\bar{k}_1}\right\}\left\{1-\varepsilon^{-1}\bar{\delta}A\right\}\Bigg\}$$

$$+b^4\Bigg\{-E^2H_1^{\,2}\frac{p_1}{p_2^2}(F\bar{\delta}\varepsilon^{-1}-\varepsilon^{-1}\bar{\delta}-aK_1)+\frac{EH_1p_1\ell_1}{p_2}(EH_1p_1+\varepsilon^{-1}F\ell_1)-\frac{H^2\pi}{4}\varepsilon^{-1}\left(\varepsilon^{-1}a^2\ell_1-\frac{Ep_1}{p_2}a\ell_1\right)$$

$$+\frac{\ell_1^2\varepsilon^{-1}}{p_2^2}(H_1-F)-\frac{H^2\pi}{4}\varepsilon^{-1}\left\{F\left\{\left(1-\frac{Ep_1}{p_2}\right)-\frac{EH_1p_1\ell_1}{p_2}\right\}(1-\varepsilon^{-1}\bar{\delta}A)+\frac{F\ell_1E}{p_2}\left(EH_1p_1+\varepsilon^{-1}\bar{\delta}A\right)\right\}+4A^2$$

$$\left(1+EH_1p_1\frac{(1+K_1)}{\bar{k}_1}\right)+\frac{2A^2FE^2}{p_2^2}(1-\varepsilon^{-1}\bar{\delta}A)\Bigg\}+b^3\Bigg\{\frac{EH_1p_1}{p_2^2}\ell_1F(H_1-1)+\frac{H^2\pi}{4}\varepsilon^{-1}\left\{EH_1p_1(2-\varepsilon^{-1}\bar{\delta}A)\right\}$$

$$\left(1-\frac{EH_1p_1}{p_2}\right)-\frac{H^2\pi}{4}\varepsilon^{-1}\left(\left\{F\ell_1\left(1-\frac{EH_1p_1}{p_2}\right)-\frac{EH_1p_1\ell_1}{p_2}\right\}(2-\varepsilon^{-1}\bar{\delta}A)\right)-\frac{(1+K_1)}{\bar{k}_1}\frac{a\ell_1}{p_2}(H_1-\ell_1)+\frac{H_1\ell_1E^2}{p_2^2}$$

$$(\ell_1+\varepsilon^{-1}H_1p_1\bar{\delta}A)+H_1^2p_1F\frac{1}{p_2}2A(1-\varepsilon^{-1}\bar{\delta}A)\Bigg\}+b^2\Bigg\{\frac{\varepsilon^{-1}H^2\pi}{4}\frac{EH_1p_1\ell_1}{p_2}(2-\varepsilon^{-1}\bar{\delta}A)+\frac{H^2\pi}{4}\varepsilon^{-1}\ell_1$$

$$\left\{\ell_1+Ep_1\left(\frac{\ell_1}{p_2}-\bar{\delta}A\varepsilon^{-1}\right)\right\}+\frac{H^2\pi}{4}H_1p_1\ell_1(1-\varepsilon^{-1}\bar{\delta}A)+\frac{H_1p_1E}{p_2}\frac{(1+K_1)}{\bar{k}_1}\Bigg\}+b\Bigg\{\varepsilon^{-1}\frac{H^2\pi}{2}EH_1p_1\ell_1FA$$

$$\left(1-\frac{Ep_1}{p_2}\right)\Bigg\}\Bigg\}\Bigg]+b^8\left\{\frac{E^2}{p_2}\left(1+H_1p_1\frac{(1+K_1)}{\bar{k}_1}(2-\varepsilon^{-1}\bar{\delta}A)-F\ell_1\varepsilon^{-1}\frac{(1+K_1)}{\bar{k}_1}\bar{\delta}Aa\right)\right\}$$

$$+b^7\left[\frac{2E^2}{p_2^2}\left\{1+EH_1p_1\frac{(1+K_1)}{\bar{k}_1}-EK_1p_1\right\}A(1-\varepsilon^{-1}\bar{\delta}A)+2Aa\varepsilon^{-1}\left\{1+EH_1p_1\frac{(1+K_1)}{\bar{k}_1}\right\}\right]$$

$$+b^6\Bigg[\frac{2A^2FE^2}{p_2^2}\left\{1+H_1p_1\frac{(1+K_1)}{\bar{k}_1}(1-\varepsilon^{-1}\bar{\delta}A)+\frac{E}{p_2^2}H_1p_1(H_1-1)\right\}+\frac{H^2\pi}{4}\varepsilon^{-1}\Bigg\{\left(\frac{EH_1p_1}{p_2}-1\right)$$

$$(2-\varepsilon^{-1}\bar{\delta}A)-\frac{EH_1p_1F}{p_2}\varepsilon^{-1}\bar{\delta}A\ell_1\Bigg\}\Bigg]+b^5\Bigg[\frac{2A^2\varepsilon^{-1}H_1}{p_2^2}(2-K_1Ep_1)+\frac{2E^2}{p_2^2}AH_1\varepsilon^{-1}(F-aK_1)+$$

$$\frac{\varepsilon^{-1}H^2\pi}{4}\left\{2AF\left(\frac{EH_1p_1}{p_2}-1\right)(2-\varepsilon^{-1}\bar{\delta}A)-\frac{EH_1p_1\ell_1}{p_2}\varepsilon^{-1}\bar{\delta}AF\right\}\Bigg]+b^4\Bigg[-\frac{2EH_1}{p_2}A^2\frac{(1+K_1)}{\bar{k}_1}+\frac{H^2\pi}{4}\varepsilon^{-1}$$

$$\left\{\left(1-\varepsilon^{-1}\bar{\delta}A\right)\left(\frac{EH_1p_1}{p_2}-1\right)+\frac{2A^2Ea}{p_2^2}K_1\left(H_1-1\right)\right\}\Bigg]+b^3\left[-\varepsilon^{-1}\frac{H^2\pi}{2}A\frac{a}{p_2}\left(H_1-1\right)+\frac{EH_1a}{p_2}\frac{\left(1+K_1\right)}{\bar{k}_1}\right]$$

$$+b^2\left[\frac{H^2\pi}{4}\varepsilon^{-1}\left(2-\varepsilon^{-1}\bar{\delta}A\right)\frac{E\,\ell_1H_1}{p_2}-\varepsilon^{-1}A^2H^2\pi\frac{aEH_1}{p_2}\left(H_1-1\right)\right]=0\,. \tag{52}$$

It is evident from the equation [52] that overstable modes will not be present for all values of parameters. For example, in the absence of coupling between spin and heat flux $\left(\bar{\delta}=0\right)$, magnetic field $\left(\vec{H}=0\right)$, $\bar{k}_1\to\infty$ and in the absence of suspended particles $\left(a=0=f=h_1\right)$, equation [52] allows only $n_i=0$ and so overstable solution will not take place if $EK_1p_1<2$.

For stationary convection**,** the marginal state is characterized by $n_i=0$; and the Rayleigh number is given by

$$R=\frac{b^3\left[\frac{\left(1+K_1\right)}{\bar{k}_1}-\varepsilon^{-1}K_1A\right]+2b^2A\frac{\left(1+K_1\right)}{\bar{k}_1}+\frac{H^2\pi}{4}\left(2A+b\right)\varepsilon^{-1}bp_2}{k^2\left\{2H_1A+b\left(1-\varepsilon^{-1}\bar{\delta}A\right)\right\}}\,. \tag{53}$$

In the absence of magnetic field intensity $\left(\mathrm{H}=0\right)$ and suspended particles $\left(a=0=f=h_1\right)$ equation [53] reduces to

$$R=\frac{b^3\left[\frac{\left(1+K_1\right)}{\bar{k}_1}-\varepsilon^{-1}K_1A\right]+2A\frac{\left(1+K_1\right)}{\bar{k}_1}b^2}{k^2\left\{2A+b\left(1-\varepsilon^{-1}\bar{\delta}A\right)\right\}}\,, \tag{54}$$

a result in good agreement with Sharma and Gupta [21].

Discussion of Results

Equation [52] has been examined numerically using the Newton–Raphson method through the software Fortran 90. We have plotted the variation of Rayleigh number with respect to wave-number using equation [51] satisfying [52] for overstable case and equation [53] for stationary case, for the fixed permissible values of the dimensionless parameters $K_1=1,\ A=0.5,\ \bar{\delta}=1,\ell_1=1,\ p_1=5,\ p_2=1,\ a=10,\ F=1.005,$
$H_1=1.01,\varepsilon=0.5,\ E=0.9,\ \ \bar{k}_1=2.$ Figures [1]–[3] correspond to three values of magnetic field intensity $H=$ 70, 100 and 120 Gauss, respectively. The graphs show that

Rayleigh number increases with increase in magnetic field intensity depicting thereby the stabilizing effect of magnetic field intensity. Moreover, the magnetic field introduces the oscillatory modes in the system. Figures [4]–[6] correspond to three values of medium permeability $\bar{k}_1 = 5$, 10 and 30. The graphs show that the Rayleigh number for the stationary convection and for the case of overstability decreases with the increase in medium permeability depicting thereby destabilizing effect of medium permeability.Figures [7]–[9] correspond to three values of micropolar coefficient $\kappa = 0.5$, 0.7 and 1.0, respectively, accounting for dynamic microrotation viscosity. The graphs show that the Rayleigh number for the stationary convection and for the case of overstability decreases with the increase in micropolar coefficient κ implying thereby the destabilizing effect of dynamic microrotation viscosity.

Figures [10]–[12] correspond to three values of micropolar coefficient γ' =1.0, 1.2 and 1.4, respectively. The graphs show that the Rayleigh number for the stationary convection and for the case of overstability decreases with the increase in micropolar coefficient γ' implying thereby the destabilizing effect of coefficient of angular viscosity, therefore micropolar coefficients have destabilizing effects on the system.

Conclusion

There is a s competition between the large enough stabilizing effect of magnetic intensity and the destabilizing effect of the micropolar coefficients. The presence of coupling between thermal and micropolar effects, magnetic field and suspended particles number density may bring overstability in the system. It is also noted from the figures [3], [4], [7] and [10] that the Rayleigh number for overstability is always less than the Rayleigh number for stationary convection, for a fixed wave-number. However, the reverse may also occur for large wave-numbers, as has been depicted in figures [1], [2], [5], [6], [8], [9], [11] and [12].

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Figures

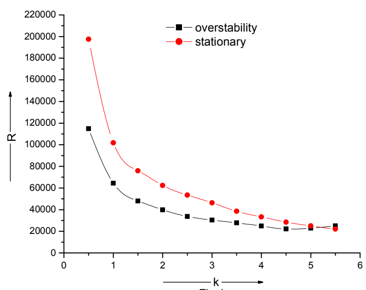

Fig.1.

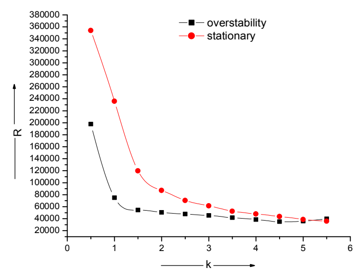

Fig.2.

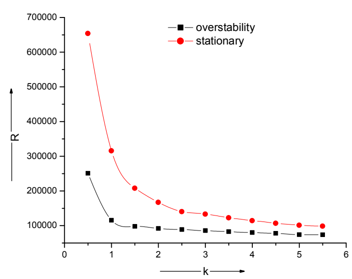

Fig.3.

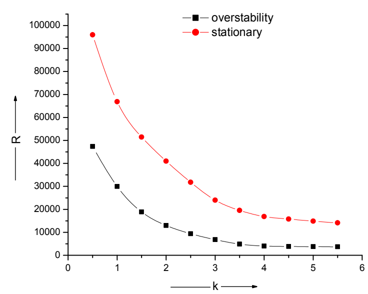

Fig.4.

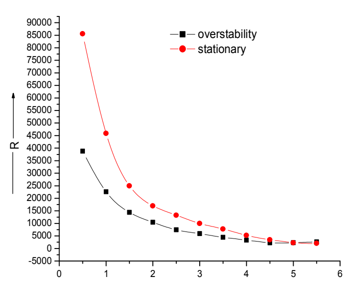

Fig.5.

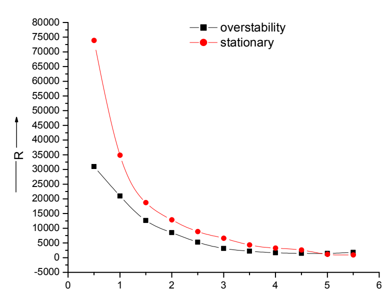

Fig.6.

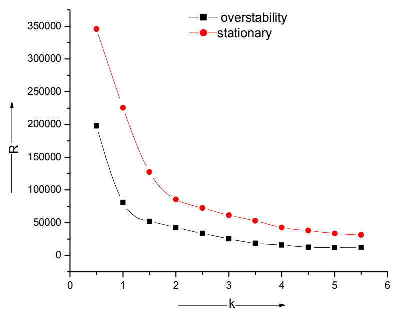

Fig.7.

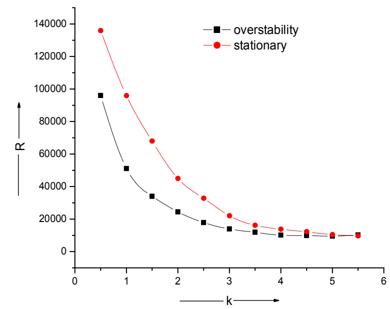

Fig.8.

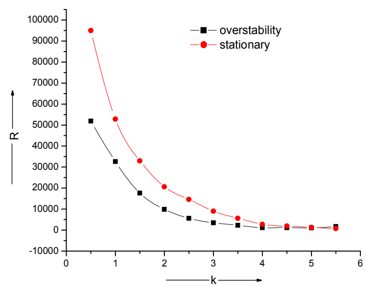

Fig.9.

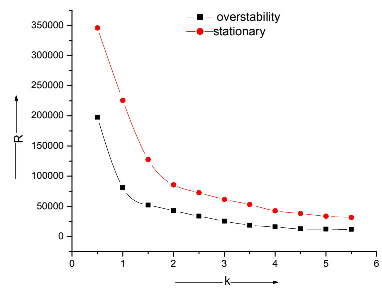

Fig.10.

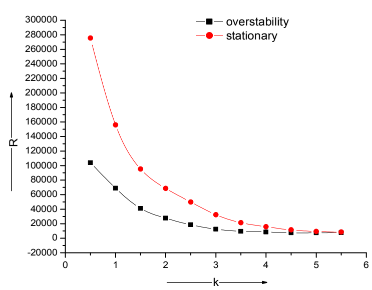

Fig.11.

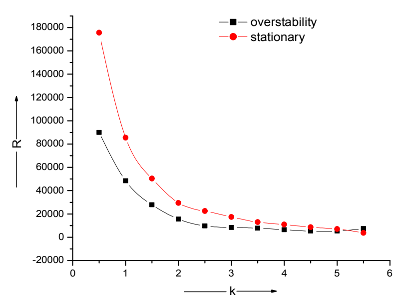

Fig.12.

Effects of porosity, Hall current and radiation on hydromagnetic flow past a heated moving vertical plate: An analysis by using Laplace transform technique

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Abstract

An investigation of the effects of porosity, Hall current and radiation on unsteady hydromagnetic free convection flow of a viscous, incompressible, electrically conducting and optically thin radiating fluid past a heated moving vertical plate embedded in porous medium is carried out. The dimensionless governing coupled; partial differential equations are solved by using Laplace transform technique. The effects of various physical parameters, encountered in the problem, on the primary and secondary fluid velocities and fluid temperature are numerically evaluated and shown through graphs, while the effects on skin-friction and rate of heat transfer are numerically evaluated and discussed with the help of tables.

Keywords: Hall current; Hydromagnetic flow; Porosity; Radiation; Free convection

Mathematical subject classification (2010): 76D05, 76D10

Introduction

The problems of MHD free convection flow in porous media have drawn attention of many researchers due to significant effect of magnetic field on the boundary layer control. On account of their varied importance, these flows have been studied by several authors. Bejan and Khair [5] investigated the vertical free convection boundary layer flow with heat and mass transfer in a porous medium. The combined heat and mass transfer effects on MHD free convective flow through porous medium have been studied by Chaudhary and Jain [6]. Singh and Kumar [17] discussed the heat and mass transfer MHD flow through porous medium. Mishra et al. [13] considered free convective MHD flow of a viscous incompressible and electrically conducting fluid past a hot vertical porous plate embedded in a porous medium. The effects of heat transfer on MHD free convective flow through porous medium with viscous dissipation have been analyzed by Poonia and Chaudhary [14].

Radiation effects on free convection flow have numerous applications in Science and engineering. Israel-Cookey et al. [10] discussed the influence of viscous dissipation and radiation on an unsteady MHD free convective flow past an infinitely long heated vertical plate in a porous medium with time dependent suction. Shankar et al. [16] analyzed radiation and mass transfer effects on MHD free convection fluid flow embedded in a porous medium with heat generation/absorption. Radiation effect on the natural convection flow of an optically thin viscous incompressible fluid near a vertical plate with ramped wall temperature in a porous medium has been studied by Das et al. [9]. Kishore et al. [12] considered the effects of thermal radiation and viscous dissipation on MHD heat and mass diffusion flow past an oscillating vertical plate embedded in a porous medium with variable surface conditions. The effects of thermal radiation and chemical reaction on MHD unsteady mass transfer flow past a semi-infinite vertical porous plate embedded in a porous medium in a slip flow regime with variable suction have been analyzed by Ahmed and Das [1]. Balla and Naikoti [4] performed a numerical analysis to study the unsteady magnetohydrodynamic convective flow of a viscous, incompressible, electrically conducting Newtonian fluid along a vertical permeable plate in the presence of a homogeneous first order chemical reaction and taking into account thermal radiation effects.

The magnetohydrodynamic free convection heat and mass transfer of a heat generating fluid past an impulsively started infinite vertical porous plate with Hall current and radiation absorption has been studied by Kinyanjui et al. [11]. Takhar et al. [18] investigated the unsteady free convective flow past an infinitely long vertical porous plate due to the combined effects of thermal and mass diffusion, magnetic field and Hall currents. Ahmed and Kalita [3] presented some model studies on the effect of Hall current on MHD convection flow. The problem of an MHD transient flow with Hall current past a uniformly accelerated horizontal porous plate in a rotating system has been discussed by Ahmed et al. [2]. Hall effects on an unsteady MHD free convective flow of a viscous incompressible electrically conducting fluid past a uniformly accelerated vertical plate in the presence of a uniform transverse applied magnetic field have been investigated by Sarkar et al. [15]. Das et al. [8] analyzed the effects of Hall currents and radiation on MHD flow of a viscous incompressible electrically conducting fluid past a moving vertical plate with variable temperature in the presence of a uniform transverse magnetic field.

Objective of the present investigation is to extend the work of Das et al. [8] and to study the effects of porosity, Hall current and radiation on unsteady hydromagnetic free convection flow of a viscous, incompressible, electrically conducting and optically thin radiating fluid past a heated moving vertical plate embedded in porous medium. The Laplace transform technique is used to solve the governing equations in order to obtain the analytical results for velocity and temperature profile, rate of heat transfer and shear stresses.

Formulation of the problem

Consider unsteady hydromagnetic free convective flow of a viscous, incompressible, electrically conducting and optically thin radiating fluid past an infinite vertical plate embedded in porous medium by taking Hall current into account. Coordinate system is chosen in such a way that x-axis is taken along the plate in the upward direction, y-axis normal to it and z-axis perpendicular to xy-plane. A uniform magnetic field of strength B_0 is applied perpendicular to the plate.

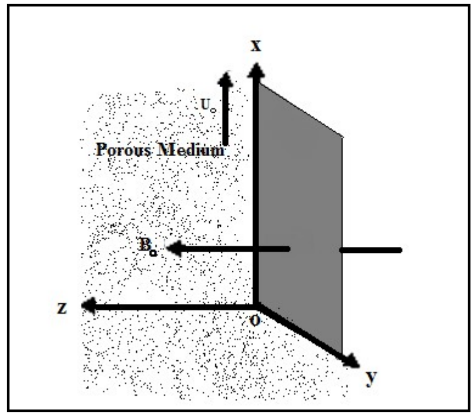

Fig. 1 Physical model of the problem

Initially, at time $t \le 0$ both the fluid and the plate are at rest and assumed to be at the same temperature T_∞. At time $t > 0$ the plate at $z = 0$ starts moving in its plane with uniform velocity U_0 and is heated with temperature $T_\infty + (T_w - T_\infty)\frac{t}{t_0}$. Since the plate is infinitely long in x and y directions, therefore all the physical quantities except pressure depend upon z and t only.

Under the usual Boussinesq approximation, equations governing the flow are given by

$$\frac{\partial w}{\partial z} = 0 \Rightarrow w = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial z^2} + g\beta\left(T - T_\infty\right) - \frac{\sigma B_0^2}{\rho\left(1 + m^2\right)}\left(u - mv\right) - \frac{\upsilon u}{K} \tag{2}$$

$$\frac{\partial v}{\partial t} = \upsilon \frac{\partial^2 v}{\partial z^2} - \frac{\sigma B_0^2}{\rho\left(1 + m^2\right)}\left(v + mu\right) - \frac{\upsilon v}{K} \tag{3}$$

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} - \frac{\partial q_r}{\partial z} \tag{4}$$

In above equations u, v, w – denote the components of velocity in the boundary layer in x, y and z directions respectively; j_x, j_y, j_z – the current density components; T – the temperature of fluid in the boundary; T_∞ – the temperature of the free stream; T_w – the temperature of the plate; t – the time; t_0 – the characteristics time; p – the fluid pressure; β – the volumetric coefficient of thermal expansion; ρ – the density of fluid; g – the acceleration due to gravity; υ – the kinematic viscosity; K – the permeability of the medium; C_p – the heat capacity of fluid at constant pressure; B_0 – the magnetic field strength; κ – the thermal conductivity of the fluid; q_r – the radiative heat flux.

The initial and boundary conditions for velocity and temperature profile are:

$$\left.\begin{aligned} &u = 0, v = 0, T = T_\infty \text{ for all } z \text{ and } t \le 0 \\ &u = U_0, v = 0, T = T_\infty + (T_w - T_\infty)\frac{t}{t_0} \text{ at } z = 0 \text{ for } t > 0 \\ &u \to 0, v \to 0, T \to T_\infty \text{ as } z \to \infty \text{ for } t > 0 \end{aligned}\right\} \qquad (5)$$

Following Cogley et al. [7], it is assumed that fluid is optically thin with a relatively low density and radiative heat flux is given by

$$\frac{\partial q_r}{\partial z} = 4(T - T_\infty)I \qquad (6)$$

where $I = \int_0^\infty K_{\lambda_w}\left(\frac{\partial e_{\lambda p}}{\partial T}\right)_w d\lambda$ (7)

In equations (7) K_λ is the absorption coefficient, λ is the wavelength, $e_{\lambda p}$ is the Plank's function and the subscript 'w' pointed out that all quantities have been evaluated at the temperature T_∞ which is the temperature of the wall at time $t \le 0$. Thus the study is limited to small difference of plate temperature to fluid temperature.

On the use of the equation (6), the equation (4) becomes

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} - 4(T - T_\infty)I \qquad (8)$$

To solve above equations, introducing following non-dimensional variables and parameters:

$$u_1 = \frac{u}{U_0},\ v_1 = \frac{v}{U_0}, \eta = \frac{zU_0}{\upsilon},\ \tau = \frac{U_0^2 t}{\upsilon},\ \theta = \frac{(T - T_\infty)}{(T_w - T_\infty)}, \Pr = \frac{\mu C_p}{\kappa},$$
$$Gr = \frac{g\beta\upsilon(T_w - T_\infty)}{U_0^3},\ M^2 = \frac{\sigma B_0^2 \upsilon}{\rho U_0^2}, K_1 = \frac{KU_0^2}{\upsilon^2}, R = \frac{4I\upsilon^2}{U_0^2\kappa} \qquad (9)$$

Using these dimensionless quantities, equations (2), (3) and (8) transform to

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial \eta^2} - \frac{M^2 \left(u_1 - m v_1\right)}{\left(1 + m^2\right)} - \frac{u_1}{K_1} + Gr\theta \tag{10}$$

$$\frac{\partial v_1}{\partial \tau} = \frac{\partial^2 v_1}{\partial \eta^2} - \frac{M^2 (v_1 + m u_1)}{\left(1 + m^2\right)} - \frac{v_1}{K_1} \tag{11}$$

$$\Pr \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} - R\theta \tag{12}$$

where M, K_1, Gr, $\Pr$ and R represents the magnetic parameter, the permeability parameter, the Grashof number, the Prandtl number and the radiation parameter respectively. In above non-dimensionalisation process, the characteristics time t_0 can be defined as $t_0 = \frac{\upsilon}{U_0^2}$

The corresponding initial and boundary conditions are

$$\left.\begin{array}{l} u_1 = 0, v_1 = 0, \theta = 0 \text{ for all } \eta \text{ and } \tau \le 0 \\ u_1 = 1, v_1 = 0, \theta = \tau \text{ at } \eta = 0 \text{ for } \tau > 0 \\ u_1 \to 0, v_1 \to 0, \theta \to 0 \text{ as } \eta \to \infty \text{ for } \tau > 0 \end{array}\right\} \tag{13}$$

Method of solution

To solve the system of equations (10) and (11), we combine these equations as follows and get

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial \eta^2} - a_1 F + Gr\theta \tag{14}$$

where $F = u_1 + \iota v_1$, $a_1 = r^2 + \frac{1}{K_1}$, $r^2 = \frac{M^2 \left(1 + \iota m\right)}{\left(1 + m^2\right)}$ and $\iota = \sqrt{-1}$

The corresponding initial and boundary conditions are

$$\left.\begin{array}{l} F = 0, \theta = 0 \text{ for all } \eta \text{ and } \tau \le 0 \\ F = 1, \theta = \tau \text{ at } \eta = 0 \text{ for } \tau > 0 \\ F \to 0, \theta \to 0 \text{ as } \eta \to \infty \text{ for } \tau > 0 \end{array}\right\} \tag{15}$$

Applying Laplace transformation and on using initial conditions, equations (12) and (14) become

$$\frac{d^2 \overline{F}}{d\eta^2} - (s + a_1)\overline{F} = -Gr\overline{\theta} \tag{16}$$

$$\frac{d^2\overline{\theta}}{d\eta^2}-\left(s\Pr+R\right)\overline{\theta}=0 \tag{17}$$

where $\overline{F}(\eta,s)=\int_0^\infty F(\eta,\tau)e^{-s\tau}d\tau$ and $\overline{\theta}(\eta,s)=\int_0^\infty \theta(\eta,\tau)e^{-s\tau}d\tau$

The corresponding boundary conditions are

$$\left.\begin{aligned}&\overline{F}(0,s)=\frac{1}{s},\ \overline{\theta}(0,s)=\frac{1}{s^2}\\&\overline{F}\to 0,\ \overline{\theta}\to 0\ \text{ as } \eta\to\infty\end{aligned}\right\} \tag{18}$$

The solution of equations (16) and (17) subject to boundary conditions (18) are given by

$$\overline{F}(\eta,s)=\begin{cases}\left(\frac{1}{s}-\frac{G_1}{b_1^2 s}-\frac{G_1}{b_1 s^2}\right)e^{-\eta\sqrt{(s+a_1)}}+\frac{G_1}{b_1^2(s-b_1)}e^{-\eta\sqrt{(s+a_1)}}+G_1\left(\frac{1}{b_1^2 s}+\frac{1}{b_1 s^2}\right)e^{-\eta\sqrt{(s\Pr+R)}}-\frac{G_1}{b_1^2(s-b_1)}e^{-\eta\sqrt{(s\Pr+R)}} & \text{when } \Pr\neq 1\\ \left(\frac{1}{s}+\frac{G_2}{s^2}\right)e^{-\eta\sqrt{(s+a_1)}}-\frac{G_2}{s^2}e^{-\eta\sqrt{(s+R)}} & \text{when } \Pr=1\end{cases} \tag{19}$$

$$\overline{\theta}(\eta,s)=\begin{cases}\frac{1}{s^2}e^{-\eta\sqrt{(s\Pr+R)}} & \text{when } \Pr\neq 1\\ \frac{1}{s^2}e^{-\eta\sqrt{(s+R)}} & \text{when } \Pr=1\end{cases} \tag{20}$$

Taking inverse Laplace transform of equations (19) and (20), we get the following expressions for velocity and temperature profile:

$$F(\eta,\tau)=\begin{cases}\left[\frac{1}{2}\left(1-\frac{G_1}{b_1^2}-\frac{G_1\tau}{b_1}-\frac{G_1\eta}{2\sqrt{a_1}b_1}\right)e^{\eta\sqrt{a_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{a_1\tau}\right)+\frac{1}{2}\left(1-\frac{G_1}{b_1^2}-\frac{G_1\tau}{b_1}+\frac{G_1\eta}{2\sqrt{a_1}b_1}\right)e^{-\eta\sqrt{a_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{a_1\tau}\right)\right]\\ +\frac{G_1e^{b_1\tau}}{2b_1^2}\left[e^{\eta\sqrt{a_1+b_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{(a_1+b_1)\tau}\right)+e^{-\eta\sqrt{a_1+b_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{(a_1+b_1)\tau}\right)\right]\\ +\left[\frac{G_1}{2}\left(\frac{1}{b_1^2}+\frac{\tau}{b_1}+\frac{\eta\sqrt{\Pr}}{2\sqrt{a_2}b_1}\right)e^{\eta\sqrt{a_2\Pr}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}+\sqrt{a_2\tau}\right)+\frac{G_1}{2}\left(\frac{1}{b_1^2}+\frac{\tau}{b_1}-\frac{\eta\sqrt{\Pr}}{2\sqrt{a_2}b_1}\right)e^{-\eta\sqrt{a_2\Pr}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}-\sqrt{a_2\tau}\right)\right]\\ -\frac{G_1e^{b_1\tau}}{2b_1^2}\left[e^{\eta\sqrt{\Pr(a_2+b_1)}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}+\sqrt{(a_2+b_1)\tau}\right)+e^{-\eta\sqrt{\Pr(a_2+b_1)}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}-\sqrt{(a_1+b_1)\tau}\right)\right]\\ \qquad\qquad\text{when } \Pr\neq 1\\ \frac{1}{2}\left[\left(1+G_2\tau+\frac{G_2\eta}{2\sqrt{a_1}}\right)e^{\eta\sqrt{a_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{a_1\tau}\right)+\left(1+G_2\tau-\frac{G_2\eta}{2\sqrt{a_1}}\right)e^{-\eta\sqrt{a_1}}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{a_1\tau}\right)\right]\end{cases}$$

$$-\frac{1}{2}\left[\left(G_2\tau+\frac{G_2\eta}{2\sqrt{R}}\right)e^{\eta\sqrt{R}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{R\tau}\right)+\frac{1}{2}\left(G_2\tau-\frac{G_2\eta}{2\sqrt{R}}\right)e^{-\eta\sqrt{R}}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{R\tau}\right)\right]$$

when Pr = 1 (21)

$$\theta(\eta,\tau)=\begin{cases}\frac{1}{2}\left[\left(\tau+\frac{\eta\sqrt{\Pr}}{2\sqrt{a_2}}\right)e^{\eta\sqrt{a_2\Pr}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}+\sqrt{a_2\tau}\right)+\left(\tau-\frac{\eta\sqrt{\Pr}}{2\sqrt{a_2}}\right)e^{-\eta\sqrt{a_2\Pr}}erfc\left(\frac{\eta\sqrt{\Pr}}{2\sqrt{\tau}}-\sqrt{a_2\tau}\right)\right] & \text{when } \Pr\neq 1\\ \frac{1}{2}\left[\left(\tau+\frac{\eta}{2\sqrt{R}}\right)e^{\eta\sqrt{R}}erfc\left(\frac{\eta}{2\sqrt{\tau}}+\sqrt{R\tau}\right)+\left(\tau-\frac{\eta}{2\sqrt{R}}\right)e^{-\eta\sqrt{R}}erfc\left(\frac{\eta}{2\sqrt{\tau}}-\sqrt{R\tau}\right)\right] & \text{when } \Pr=1\end{cases}\quad (22)$$

Some important characteristics of flow

From the velocity field equation (21), the expression for the dimensionless shear stress (τ^*) at the plate is given by

$$\tau^*=\left(\frac{\partial F}{\partial\eta}\right)_{\eta=0}=\begin{cases}\left[\frac{G_1}{2b_1\sqrt{a_1}}-\sqrt{a_1}\left(1-\frac{G_1}{b_1^2}-\frac{G_1\tau}{b_1}\right)\right]erf\left(\sqrt{a_1\tau}\right)-\frac{1}{\sqrt{\pi t}}\left(1-\frac{G_1}{b_1^2}-\frac{G_1\tau}{b_1}\right)e^{-a_1\tau}\\ -\frac{G_1e^{b_1\tau}}{b_1^2}\left[\sqrt{(a_1+b_1)}erf\left(\sqrt{(a_1+b_1)\tau}\right)+\frac{1}{\sqrt{\pi\tau}}e^{-(a_1+b_1)\tau}\right]-\left[\frac{G_1\sqrt{\Pr}}{2\sqrt{a_2}b_1}+\left(\frac{G_1}{b_1^2}+\frac{G_1\tau}{b_1}\right)\sqrt{a_2\Pr}\right]erf\left(\sqrt{a_2\tau}\right)\\ -\frac{\sqrt{\Pr}}{\sqrt{\pi\tau}}\left(\frac{G_1}{b_1^2}+\frac{G_1\tau}{b_1}\right)e^{-(a_2\tau)}+\frac{G_1e^{b_1\tau}}{b_1^2}\left[\sqrt{\Pr(a_2+b_1)}erf\left(\sqrt{(a_2+b_1)\tau}\right)+\frac{\sqrt{\Pr}}{\sqrt{\pi\tau}}e^{-(a_2+b_1)\tau}\right]\\ \text{when } \Pr\neq 1\\ -\left[\frac{G_2}{2\sqrt{a_1}}+(1+G_2\tau)\sqrt{a_1}\right]erf\left(\sqrt{a_1\tau}\right)-\frac{(1+G_2\tau)}{\sqrt{\pi\tau}}e^{-a_1\tau}+\left[\frac{G_2}{2\sqrt{R}}+G_2\tau\sqrt{R}\right]erf\left(\sqrt{R\tau}\right)+G_2\sqrt{\frac{\tau}{\pi}}e^{-R\tau}\\ \text{when } \Pr=1\end{cases}\quad (23)$$

From the temperature field equation (22) the expression for the dimensionless rate of heat transfer coefficient (*Nu*) at the plate is given by

$$Nu=-\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}=\begin{cases}\left(\tau\sqrt{a_2\Pr}+\frac{\sqrt{\Pr}}{2\sqrt{a_2}}\right)erf\left(\sqrt{a_2\tau}\right)+\frac{\sqrt{\tau\Pr}}{\sqrt{\pi}}e^{-a_2\tau} & \text{when } \Pr\neq 1\\ \left(\tau\sqrt{a_2}+\frac{1}{2\sqrt{a}}\right)erf\left(\sqrt{a_2\tau}\right)+\sqrt{\frac{\tau}{\pi}}e^{-a_2\tau} & \text{when } \Pr=1\end{cases}\quad (24)$$

Results and Discussion

In order to analyze the effects of Hall current, thermal buoyancy force, thermal radiation and permeability of the medium on the flow field in the boundary layer region, numerical computations have been carried out for variations in the governing parameters such as the Hall parameter (m), the Grashof number (Gr), the radiation parameter (R), the permeability parameter (K_1), the Prandtl number ($\Pr$) and the magnetic field parameter (M). For illustration of these results, numerical values are plotted in figures (2-8). Our results agree with the results of Das et al. [8] in the absence of porous medium.

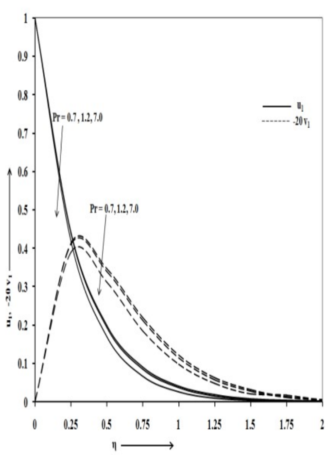

Fig.2 Velocity profile for different values of the Prandtl number ($\Pr$) when $Gr = 5$, $M = 2$, $R = 4$, $m = 0.4$, $K_1 = 0.1$, $\tau = 0.5$

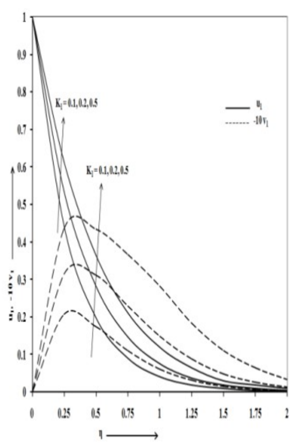

Fig.3 Velocity profile for different values of the permeability parameter (K_1) when $Gr = 5$, $M = 2$, $R = 4$, $m = 0.4$, $\Pr = 0.7$, $\tau = 0.5$

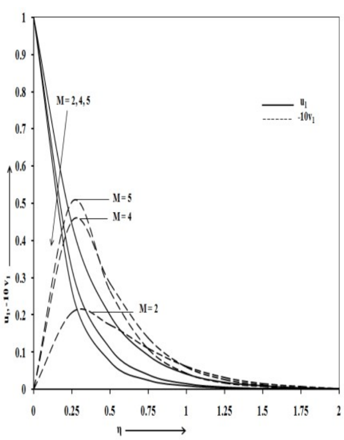

Fig.4 Velocity profile for different values of the magnetic parameter (M) when $Gr = 5$, $K_1 = 0.1$, $R = 4$, $m = 0.4$, $\Pr = 0.7$, $\tau = 0.5$

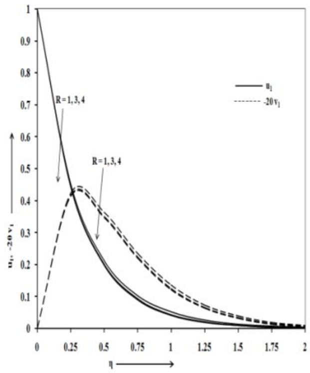

Fig.5 Velocity profile for different values of the radiation parameter (R) when $Gr = 5$, $K_1 = 0.1$, $M = 2$, $m = 0.4$, $\Pr = 0.7$, $\tau = 0.5$

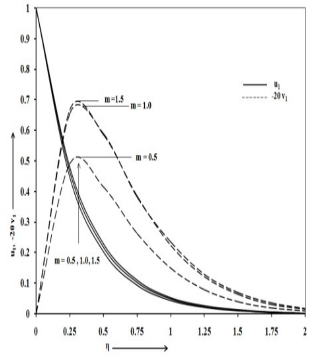

Fig.6 Velocity profile for different values of the Hall parameter (m) when $Gr = 5$, $K_1 = 0.1$, $M = 2$, $R = 4$, $\Pr = 0.7$, $\tau = 0.5$

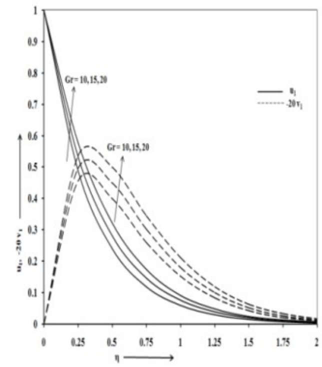

Fig.7 Velocity profile for different values of the Grashof number (Gr) when $m = 0.4$, $K_1 = 0.1$, $M = 2$, $R = 4$, $\Pr = 0.7$, $\tau = 0.5$

From fig.2 it is observed that the primary velocity and the magnitude of the secondary velocity decrease with an increase in the Prandtl number. The fluids with high value of the Prandtl number have greater viscosity, which makes the fluid thick and hence move slowly. It is found from fig.3 that both the primary and the secondary fluid velocities increase with an increase in the permeability parameter (K_1) in the boundary layer region. This is due to the fact that the presence of a porous medium decreases the resistance to flow. Fig.4 shows that the primary velocity is diminished and the secondary velocity is increased when the magnetic parameter (M) is increased. When a transverse magnetic field is applied then the Lorentz force acts in a direction opposite to the flow which tends to resist the flow thereby reducing the primary velocity. On the other hand, for the secondary flow this force acts as an aiding force. This will serve to accelerate the secondary velocity. Fig. 5 displays the effect of the radiation parameter on the primary and the secondary velocities. It is noticed that increase in the radiation parameter decreases primary and secondary velocities. Increase in the radiation emission reduces the rate of heat transfer through the fluid, which results in the decrease in temperature in the boundary layer. The velocity decreases due to reduction in buoyancy forces associated with the decreased temperature. The effect of Hall current on the primary and the secondary velocities is depicted through fig. 6. It is inferred from the figure that the Hall current promotes the flow along the plate. This is because, the Hall current reduces the resistance offered by the Lorentz force. From fig.7 it is observed that the primary velocity and the magnitude of the secondary velocity increase with an increase in Grashof number. There is a rise in the fluid velocity due to the enhancement of thermal buoyancy force

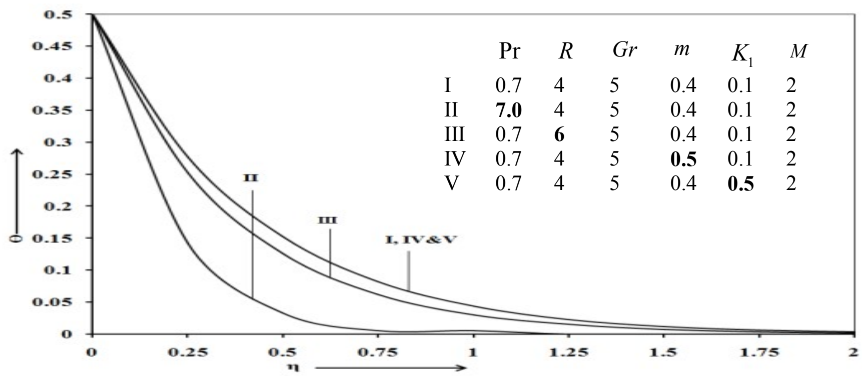

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Fig.8 Temperature profile for different values of Pr , R , m and K_1 for $\tau = 0.5$

Fig.8 reveals that fluid temperature in the boundary layer decreases on increasing the Prandtl number and the radiation parameter. This result qualitatively agrees with expectations, since thermal diffusivity decreases with increase in the Prandtl number and the effect of radiation is to decrease the rate of energy transport to the fluid, thereby decreasing the temperature of the fluid.

Table 1 Numerical values of shear stress $\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} = -\left(\tau_x + \iota\tau_y\right)$ at the plate

Pr	Gr	R	M	K_1	m	$-\tau_x$	$-\tau_y$
0.7	5	2	2	0.1	0.4	3.2515	0.19956
7.0	5	2	2	0.1	0.4	3.3921	0.19268
0.7	10	2	2	0.1	0.4	2.831	0.21135
0.7	5	4	2	0.1	0.4	3.2794	0.19816
0.7	5	2	4	0.1	0.4	4.5585	0.58808
0.7	5	2	2	0.5	0.4	1.8418	0.31122
0.7	5	2	2	0.1	1.0	3.0433	0.30642

Table 2 Numerical values of Rate of heat transfer coefficient $Nu = -\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0}$ at the plate

R	Pr	τ	Nu
4	0.7	0.5	1.1744
6	0.7	0.5	2.3065
4	7.0	0.5	1.3676
4	0.7	1.0	2.1750

Numerical results of the shear stresses due to the primary and the secondary flow at the plate are expressed in the table 1 for different values of governing parameters. From table 1 it is observed that the absolute value of shear stress τ_x increases with increase in the Prandtl number, the radiation parameter and the magnetic parameter but decreases with the Grashof number, the permeability parameter and the Hall parameter whereas the absolute value of the shear stress τ_y decreases with an increase in the Prandtl number and the radiation parameter whereas increases with remaining parameters. From table 2 it is noticed the rate of heat transfer at the plate increases with an increase in the radiation parameter, the Prandtl number and time.

Conclusions

From the study the following conclusions are drawn:

- Porosity, Hall current and thermal buoyancy forces promote the flow throughout the boundary layer region by accelerating both the primary and secondary velocity components.
- Primary and secondary velocity components decrease in the presence of thermal radiation.
- Applied magnetic field retards the primary flow and supports the secondary flow.
- Thermal buoyancy forces, Hall current and porosity reduce the shear stress at the plate.
- There is an enhancement in rate of heat transfer at the plate with thermal radiation and thermal diffusion.

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Characterization of Thermosolutal Convection in Couple-Stress Fluid in a Porous Medium in the Presence of a Magnetic Field

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ABSTRACT

Thermosolutal instability of Veronis(1965) type in a couple-stress fluid in the presence of uniform vertical magnetic field in a porous medium is considered. Following the linearized stability theory and normal mode analysis, the paper mathematically established the condition for characterizing the oscillatory motions which may be neutral or unstable, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid. It is proved analytically that all non-decaying slow motions starting from rest, in a couple-stress fluid of infinite horizontal extension and finite vertical depth in a porous medium, which is acted upon by uniform vertical magnetic field opposite to force field of gravity and a constant vertical adverse temperature gradient, are necessarily non-oscillatory, in the regime established, the result is important since the exact solutions of the problem investigated are not obtainable in closed form, for any arbitrary combination of free and rigid boundaries. A similar characterization theorem is also established for Stern (1960) type of configuration.

Key Words: Thermosolutal convection; couple-stress Fluid; Magnetic Field; Rayleigh number; Chandrasekhar number.

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1. INTRODUCTION

A detailed account of the theoretical and experimental study of the onset of thermal instability in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [1] and the Boussinesq approximation has been used throughout, which states that the density changes are disregarded in all other terms in the equation of motion, except in the external force term. The formation and derivation of the basic equations of a layer of fluid heated from below in a porous medium, using the Boussinesq approximation, has been given in a treatise by Joseph [2]. When a fluid permeates through an isotropic and homogeneous porous medium, the gross effect is represented by Darcy's law. The study of layer of fluid heated from below in porous media is motivated both theoretically and by its practical applications in engineering. Among the applications in engineering disciplines one can name the food processing industry, the chemical processing industry, solidification, and the centrifugal casting of metals. The development of geothermal power resources has increased general interest in the properties of convection in a porous medium. The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been considered by Veronis [3]. Double-diffusive convection problems arise in oceanography (salt fingers occur in the ocean when hot saline water overlies cooler fresher water which believed to play an important role in the mixing of properties in several regions of the ocean), limnology and engineering. The migration of moisture in fibrous insulation, bio/chemical contaminants transport in environment, underground disposal of nuclear wastes, magmas, groundwater, high quality crystal production and production of pure medication are some examples where double-diffusive convection is involved. Examples of particular interest are provided by ponds built to trap solar heat Tabor and Matz [4]and some Antarctic lakes Shirtcliffe [5]. The physics is quite similar in the stellar case in that helium acts like salt in raising the density and in diffusing more slowly than heat. The conditions under which convective motions are important in stellar atmospheres are usually far removed from consideration of a single component fluid and rigid boundaries, and therefore it is desirable to consider a fluid acted on by a solute gradient and free boundaries.

The flow through porous media is of considerable interest for petroleum engineers, for geophysical fluid dynamists and has importance in chemical technology and industry. An example in the geophysical context is the recovery of crude oil from the pores of reservoir rocks. Among the application in engineering disciplines one can find the food processing industry, chemical processing industry, solidification and centrifugal casting of metals. Such flows has shown their great importance in petroleum engineering to study the movement of natural gas, oil and water through the oil reservoirs; in chemical engineering for filtration and purification processes and in the field of agriculture engineering to study the underground water resources, seepage of water in river beds. The problem of thermosolutal convection in fluids in a porous medium is of importance in geophysics, soil sciences, ground water hydrology and astrophysics. The study of thermosolutal convection in fluid saturated porous media has diverse practical applications, including that related to the materials processing technology, in particular, the melting and solidification of binary alloys. The development of geothermal power resources has increased general interest in the properties of convection in porous media.

The scientific importance of the field has also increased because hydrothermal circulation is the dominant heat-transfer mechanism in young oceanic crust Lister [6]. Generally it is accepted that comets consists of a dusty 'snowball' of a mixture of frozen gases which in the process of their journey changes from solid to gas and vice - versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in the astrophysical context Mc Donnel [7]. The effect of a magnetic field on the stability of such a flow is of interest in geophysics, particularly in the study of Earth's core where the Earth's mantle, which consists of conducting fluid, behaves like a porous medium which can become convectively unstable as a result of differential diffusion. The other application of the results of flow through a porous medium in the presence of a magnetic field is in the study of the stability of a convective flow in the geothermal region. Also the magnetic field in double-diffusive convection has its importance in the fields of engineering, for example, MHD generators and astrophysics particularly in explaining the properties of large stars with a helium rich core. Stommel and Fedorov [8] and Linden [9] have remarked that the length scales characteristics of double-diffusive convective layers in the ocean may be sufficiently large that the Earth's rotation might be important in their formation. Moreover, the rotation of the Earth distorts the boundaries of a hexagonal convection cell in a fluid through a porous medium and the distortion plays an important role in the extraction of energy in the geothermal regions. Brakke [10] explained a double - diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose. Nason et al. [11] found that this instability, which is deleterious to certain biochemical separations, can be suppressed by rotation in the ultracentrifuge.

The theory of couple-stress fluid has been formulated by Stokes [12]. One of the applications of couple-stress fluid is its use to the study of the mechanisms of lubrications of synovial joints, which has become the object of scientific research. A human joint is a dynamically loaded bearing which has articular cartilage as the bearing and synovial fluid as the lubricant. When a fluid film is generated, squeeze - film action is capable of providing considerable protection to the cartilage surface. The shoulder, ankle, knee and hip joints are the loaded – bearing synovial joints of the human body and these joints have a low friction coefficient and negligible wear. Normal synovial fluid is a viscous, non-Newtonian fluid and is clear or yellowish. According to the theory of Stokes [12], couple-stresses appear in noticeable magnitudes in fluids with very large molecules. Since the long chain hyaluronic acid molecules are found as additives in synovial fluids, Walicki and Walicka [13] modeled the synovial fluid as a couple-stress fluid. The synovial fluid is the natural lubricant of joints of the vertebrates. The detailed description of the joint lubrication has very important practical implications. Practically all diseases of joints are caused by or connected with malfunction of the lubrication. The efficiency of the physiological joint lubrication is caused by several mechanisms. The synovial fluid is due to its content of the hyaluronic acid, a fluid of high viscosity, near to gel. Goel et al. [14] have studied the hydromagnetic stability of an unbounded couple-stress binary fluid mixture under rotation with vertical temperature and concentration gradients. Sharma et al. [15] have considered a couple - stress fluid with suspended particles heated from below. In another study, Sunil et al. [16] have considered a couple- stress fluid heated from below in a porous medium in the presence of a magnetic field and rotation. Kumar et al. [17] have considered the thermal instability of a layer of couple-stress fluid acted on by a uniform rotation, and have found that for stationary convection the rotation has a stabilizing effect whereas couple-stress has both stabilizing and destabilizing effects.

Pellow and Southwell [18] proved the validity of PES for the classical Rayleigh-Bénard convection problem. Banerjee et al [19] gave a new scheme for combining the governing equations of thermohaline convection, which is shown to lead to the bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries and, Banerjee and Banerjee [20] established a criterion on characterization of non-oscillatory motions in hydrodynamics which was further extended by Gupta et al [21]. However no such result existed for non-Newtonian fluid configurations in general and in particular, for Rivlin-Ericksen viscoelastic fluid configurations. Banyal [22] have characterized the oscillatory motions in couple-stress fluid.

Keeping in mind the importance of non-Newtonian fluids, as stated above, this article attempts to study couple-stress fluid of Veronis and Stern type configuration in the presence of uniform magnetic field in a porous medium, and it has been established that the onset of instability in a couple-stress fluid in a porous medium Veronis type configuration, cannot manifest itself as oscillatory motions of growing amplitude if the Thermosolutal Rayliegh number R_s, the Chandrasekhar number Q, the magnetic Prandtl number p_2, the thermosolutal Prandtl number p_3, the medium permeability P_l, the porosity ε and the viscoelasticity parameter F satisfy the inequality $R_s \leq 4\pi^4\left\{\frac{F}{P_l}+\frac{1}{E^{/}p_3\varepsilon}\left(1-\frac{Qp_2}{\pi^2}\right)\right\}$, for all wave numbers and for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid. A similar characterization theorem is also proved for Stern [23] type of configuration, for all wave numbers and for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid.

2. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we consider an infinite, horizontal, incompressible couple-stress fluid layer of thickness d, heated and soluted from below so that, the temperatures, densities and solute concentrations at the bottom surface z = 0 are T_0, ρ_0 and C_0 and at the upper surface z = d are T_d, ρ_d and C_d respectively, and that a uniform temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ and a uniform solute gradient $\beta^{/}\left(=\left|\frac{dC}{dz}\right|\right)$ are maintained. The gravity field $\vec{g}(0,0,-g)$ and a uniform vertical magnetic field $\vec{H}(0,0,H)$ pervade the system. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity $\in$ and medium permeability k_1.

Let p, ρ, T, C, α, $\alpha^{/}$, g, η, μ_e and $\vec{q}(u,v,w)$ denote respectively, the fluid pressure, density, temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficient of expansion, gravitational acceleration, resistivity, magnetic permeability and fluid velocity. The equations expressing the conservation of momentum, mass, temperature, solute concentration and equation of state of couple-stress fluid (Chandrasekhar[1]; Joseph[2]; Stokes[12]) are

$$\frac{1}{\in}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\in}(\vec{q}.\nabla)\vec{q}\right]=-\left(\frac{1}{\rho_0}\right)\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu-\frac{\mu^{/}}{\rho_0}\nabla^2\right)\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{H}\right)\times\vec{H}, \tag{1}$$

$$\nabla.\vec{q}=0, \tag{2}$$

$$E\frac{\partial T}{\partial t}+\left(\vec{q}.\nabla\right)T=\kappa\nabla^2 T, \tag{3}$$

$$E^{/}\frac{\partial C}{\partial t}+\left(\vec{q}.\nabla\right)C=\kappa^{/}\nabla^2 C, \tag{4}$$

$$\rho = \rho_0\,[1-\alpha\,(T-T_0)+\alpha^{/}\,(C-C_0)], \tag{5}$$

Where the suffix zero refers to values at the reference level z = 0 and in writing equation (1), use has been made of Boussinesq approximation. Here E = $\in+(1-\in)\left(\frac{\rho_s C_s}{\rho_0 C_i}\right)$ is a constant and $E^{/}$ is a constant analogous to E but corresponding to solute rather that heat; ρ_s, ρ_o, C_s and C_i stand for density and heat capacity of solid (porous matrix) material and fluid, respectively. The magnetic permeability μ_e, the kinematic viscosityν, couple-stress viscosity $\mu^/$, the thermal diffusivity κ and the solute diffusivity $\kappa^/$ are all assumed to be constants.

The Maxwell's equations yield

$$\in \frac{d\vec{H}}{dt} = \left(\vec{H}\cdot\nabla\right)\vec{q} + \in \eta \nabla^2 \vec{H}\,, \tag{6}$$

and $\nabla \cdot \vec{H} = 0,$ (7)

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \in^{-1} \vec{q}\cdot\nabla$ stands for the convective derivative.

The steady state solution is

$\vec{q}(u,v,w) = (0,0,0)$, $T = T_0 - \beta z$, $C = C_0 - \beta^/ z$,

$\rho = \rho_0 (1+ \alpha\beta z - \alpha^/\beta^/ z)$. (8)

Here we use linearized stability theory and normal mode analysis method. Consider a small perturbation on the steady state solution, and let δp, $\delta\rho$, θ, γ , $\vec{h}(h_x, h_y, h_z)$ and $\vec{q}(u,v,w)$ denote, respectively, the perturbations in pressure p, density ρ, temperature T, solute concentration C, magnetic field $\vec{H}(0,0,0)$ and velocity $\vec{q}(0,0,0)$. The change in density$\delta\rho$, caused mainly by the perturbations θ and γ in temperature and concentration, is given by

$$\delta\rho = - \rho_0 (\alpha\theta - \alpha^/\gamma). \tag{9}$$

Then the linearized perturbation equations become

$$\frac{1}{\in}\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_0}\nabla\delta p - \vec{g}(\alpha\theta - \alpha^/\gamma) - \frac{1}{k_1}\left(\nu - \frac{\mu^/}{\rho_0}\nabla^2\right)\vec{q} + \frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{h}\right)\times\vec{H}\,, \tag{10}$$

$$\nabla.\vec{q} = 0, \tag{11}$$

$$E\frac{\partial\theta}{\partial t} = \beta w + \kappa\nabla^2\theta\,, \tag{12}$$

$$E^/\frac{\partial\gamma}{\partial t} = \beta^/ w + \kappa^/\nabla^2\gamma\,, \tag{13}$$

$$\in \frac{\partial \vec{h}}{\partial t} = \left(\vec{H} \cdot \nabla \right) \vec{q} + \in \eta \nabla^2 \vec{h} , \tag{14}$$

and $\nabla . \vec{h} = 0.$ (15)

3. NORMAL MODES ANALYSIS

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z, \gamma] = [W(z), \Theta(z), K(z), \Gamma(z)] \exp(ik_x x + ik_y y + nt), \tag{16}$$

where k_x, k_y are the wave numbers along the x- and y- directions respectively, k= ($\sqrt{k_x^2 + k_y^2}$) is the resultant wave number and n is the growth rate which is, in general, a complex constant. $W(z), K(z), \Theta(z)$ and $\Gamma(z)$ are the functions of z only.

Using (16), equations (10)-(15), within the framework of Boussinesq approximations, in the non-dimensional form transform to

$$\left(D^2 - a^2\right)\left[\frac{F}{P_l}\left(D^2 - a^2\right) - \left(\frac{\sigma}{\varepsilon} + \frac{1}{P_l}\right)\right]W = Ra^2\Theta - R_s a^2 \Gamma - Q\left(D^2 - a^2\right)DK , \tag{17}$$

$$\left(D^2 - a^2 - Ep_1\sigma\right)\Theta = -W , \tag{18}$$

$$\left(D^2 - a^2 - E^{/} p_3\sigma\right)\Gamma = -W , \tag{19}$$

And

$$\left(D^2 - a^2 - p_2\sigma\right)K = -DW , \tag{20}$$

Where we have introduced new coordinates (x', y', z') = (x/d, y/d, z/d) in new units of length d and $D = d / dz'$. For convenience, the dashes are dropped hereafter. Also we have substituted $a = kd, \sigma = \frac{nd^2}{\nu}$, $p_1 = \frac{\nu}{\kappa}$, is the thermal Prandtl number;, $p_3 = \frac{\nu}{\kappa'}$ is the thermosolutal Prandtl number; $p_2 = \frac{\nu}{\eta}$ is the magnetic Prandtl number; $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability, $F = \frac{\mu' /(\rho_0 d^2)}{\nu}$, is the dimensionless couple-stress parameter; $R = \frac{g\alpha\beta d^4}{\kappa\nu}$, is the thermal Rayleigh number; $R_s = \frac{g\alpha'\beta' d^4}{\kappa'\nu}$ is the thermosolutal Rayleigh number and $Q = \frac{\mu_e H^2 d^2}{4\pi\rho_0 \nu\eta\varepsilon}$, is the Chandrasekhar number. Also we have Substituted $W = W_{\oplus}$, $\Theta = \frac{\beta d^2}{\kappa}\Theta_{\oplus}$, $\Gamma = \frac{\beta' d^2}{\kappa'}\Gamma_{\oplus}$, $K = \frac{Hd}{\varepsilon\eta}K_{\oplus}$ and $D_{\oplus} = dD$, and dropped $(\oplus)$ for convenience.

We now consider the cases where the boundaries are rigid-rigid or rigid-free or free-rigid or free-free at z = 0 and z = 1 respectively, as the case may be, and are perfectly conducting maintained at constant temperature and solute concentration. Then the perturbations in the temperature and solute concentration are zero at the boundaries. The appropriate boundary conditions with respect to which equations (17) -- (20), must possess a solution are

$W = 0 = \Theta = \Gamma$, on both the horizontal boundaries,

DW = 0, on a rigid boundary,

$D^2W = 0$, on a dynamically free boundary,

K = 0, on both the boundaries as the regions outside the fluid

Are perfectly conducting, (21)

Equations (17)-(20), along with boundary conditions (21), pose an eigenvalue problem for σ and we wish to characterize σ_i, when $\sigma_r \geq 0$.

We first note that since W, K, Θ and Γ satisfy $W(0) = 0 = W(1)$, $K(0) = 0 = K(1)$, $\Theta(0) = 0 = \Theta(1)$ and $\Gamma(0) = 0 = \Gamma(1)$ in addition to satisfying to governing equations and hence we have from the Rayleigh-Ritz inequality Schultz [24]

$$\int_0^1 |DW|^2 dz \geq \pi^2 \int_0^1 |W|^2 dz, \int_0^1 |DK|^2 dz \geq \pi^2 \int_0^1 |K|^2 dz, \int_0^1 |D\Theta|^2 dz \geq \pi^2 \int_0^1 |\Theta|^2 dz,$$

and $\int_0^1 |D\Gamma|^2 dz \geq \pi^2 \int_0^1 |\Gamma|^2 dz$, (22)

Further, for $W(0) = 0 = W(1)$, Banerjee et al [25] have shown that

$$\int_0^1 |D^2W|^2 dz \geq \pi^2 \int_0^1 |DW|^2 dz. \tag{23}$$

4. MATHEMATICAL ANALYSIS

We prove the following lemma:

Lemma 1: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 \left(|D\Theta|^2 + a^2 |\Theta|^2 \right) dz \le \frac{1}{\pi^2 (\pi^2 + a^2)} \int_0^1 |DW|^2 \, dz \, .$$

Proof: Multiplying equation (18) by Θ^* (the complex conjugate of Θ), integrating by parts each term of the resulting equation on the right hand side for an appropriate number of times and making use of boundary condition on Θ namely $\Theta(0) = 0 = \Theta(1)$, it follows that

$$\int_0^1 \left\{ |D\Theta|^2 + a^2 |\Theta|^2 \right\} dz + E\sigma_r p_1 \int_0^1 |\Theta|^2 \, dz = \text{Real part of} \left\{ \int_0^1 \Theta^* W dz \right\},$$

$$\le \left| \int_0^1 \Theta^* W dz \right| \le \int_0^1 |\Theta^* W| dz \le \int_0^1 |\Theta^*| |W| dz \, ,$$

$$\le \int_0^1 |\Theta| |W| dz \le \left\{ \int_0^1 |\Theta|^2 \, dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |W|^2 \, dz \right\}^{\frac{1}{2}}, \tag{24}$$

(Utilizing Cauchy-Schwartz-inequality),

So that by using inequality (22) and the fact that $\sigma_r \ge 0$, we obtain from the above that

$$(\pi^2 + a^2) \int_0^1 |\Theta|^2 \, dz \le \left\{ \int_0^1 |\Theta|^2 \, dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |W|^2 \, dz \right\}^{\frac{1}{2}},$$

And thus, we get

$$\left\{ \int_0^1 |\Theta|^2 \, dz \right\}^{\frac{1}{2}} \le \frac{1}{(\pi^2 + a^2)} \left\{ \int_0^1 |W|^2 \, dz \right\}^{\frac{1}{2}}, \tag{25}$$

Since $\sigma_r \ge 0$ and $p_1 \rangle 0$, hence inequality (24) on utilizing (25) and (22), gives

$$\int_0^1 \left(|D\Theta|^2 + a^2 |\Theta|^2 \right) dz \le \frac{1}{\pi^2 (\pi^2 + a^2)} \int_0^1 |DW|^2 \, dz \, , \tag{26}$$

This completes the proof of lemma 1.

Lemma 2: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 \left(|D\Gamma|^2 + a^2|\Gamma|^2\right)dz \le \frac{1}{\pi^2(\pi^2 + a^2)}\int_0^1 |DW|^2\, dz\,.$$

Proof: Multiplying equation (19) by Γ^* (the complex conjugate of Γ), integrating by parts each term of the resulting equation on the right hand side for an appropriate number of times and making use of boundary condition on Γ namely $\Gamma(0) = 0 = \Gamma(1)$, it follows that

$$\int_0^1 \left\{|D\Gamma|^2 + a^2|\Gamma|^2\right\}dz + E^{/}\sigma_r p_3 \int_0^1 |\Gamma|^2\, dz = \text{Real part of}\left\{\int_0^1 \Gamma^* W dz\right\},$$

$$\le \left|\int_0^1 \Gamma^* W dz\right| \le \int_0^1 |\Gamma^* W| dz \le \int_0^1 |\Gamma^*||W| dz\,,$$

$$\le \int_0^1 |\Gamma||W| dz \le \left\{\int_0^1 |\Gamma|^2\, dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |W|^2\, dz\right\}^{\frac{1}{2}}, \tag{27}$$

(Utilizing Cauchy-Schwartz-inequality),

So that by using inequality (22) and the fact that $\sigma_r \ge 0$, we obtain from the above that

$$(\pi^2 + a^2)\int_0^1 |\Gamma|^2\, dz \le \left\{\int_0^1 |\Gamma|^2\, dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |W|^2\, dz\right\}^{\frac{1}{2}},$$

And thus, we get

$$\left\{\int_0^1 |\Gamma|^2\, dz\right\}^{\frac{1}{2}} \le \frac{1}{(\pi^2 + a^2)}\left\{\int_0^1 |W|^2\, dz\right\}^{\frac{1}{2}}, \tag{28}$$

Since $\sigma_r \ge 0$ and $p_1 \rangle 0$, hence inequality (27) on utilizing (28) and (22), gives

$$\int_0^1 \left(|D\Gamma|^2 + a^2|\Gamma|^2\right)dz \le \frac{1}{\pi^2(\pi^2 + a^2)}\int_0^1 |DW|^2\, dz\,, \tag{29}$$

This completes the proof of lemma 2.

Lemma 3: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 \left\{ |DK|^2 + a^2 |K|^2 \right\} dz \le \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz$$

Proof: Multiplying equation (20) by K^* (the complex conjugate of K), integrating by parts each term of the resulting equation on the left hand side for an appropriate number of times and making use of boundary conditions on K namely $K(0) = 0 = K(1)$, it follows that

$$\int_0^1 \left\{ |DK|^2 + a^2 |K|^2 \right\} dz + \sigma_r p_2 \int_0^1 |K|^2 \, dz = \text{Real part of} \left\{ \int_0^1 K^* DW dz \right\} \le \left| \int_0^1 K^* DW dz \right|$$

$$\le \int_0^1 |K^* DW| dz, \le \int_0^1 |K^*| |DW| dz \le \int_0^1 |K| |DW| dz \le \left\{ \int_0^1 |K|^2 \, dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |DW|^2 \, dz \right\}^{\frac{1}{2}}, \tag{30}$$

(Utilizing Cauchy-Schwartz-inequality),

Inequality (30) on utilizing (22), gives

$$\left\{ \int_0^1 |K|^2 \, dz \right\}^{\frac{1}{2}} \le \frac{1}{\pi^2} \left\{ \int_0^1 |DW|^2 \, dz \right\}^{\frac{1}{2}}, \tag{31}$$

Since $\sigma_r \ge 0$ and $p_2 \rangle 0$, hence inequality (30) on utilizing (31), give

$$\int_0^1 \left(|DK|^2 + a^2 |K|^2 \right) dz \le \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz, \tag{32}$$

This completes the proof of lemma 3**.**

Now we prove the following theorems:

Theorem 1: If $R \rangle 0, R_s \rangle 0$ $F \rangle 0$, $Q \rangle 0$, $P_l \rangle 0$, $p_1 \rangle 0$, $p_2 \rangle 0$, $\sigma_r \ge 0$, $\sigma_i \ne 0$, $\frac{Qp_2}{\pi^2} \le 1$ and $R_s \ge R$ then the necessary condition for the existence of non-trivial solution (W, Θ, Γ, K) of equations (17) – (20), together with boundary conditions (21) is that

$$R_s \rangle 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E^{/} p_3 \varepsilon} \left(1 - \frac{Qp_2}{\pi^2} \right) \right\}.$$

Proof: Multiplying equation (17) by W^* (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$\frac{F}{P_l}\int_0^1 W^*\left(D^2-a^2\right)^2 W dz-\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\right)\int_0^1 W^*\left(D^2-a^2\right)W dz$$

$$= Ra^2\int_0^1 W^*\Theta dz - R_s a^2\int_0^1 W^*\Gamma dz - Q\int W^* D(D^2-a^2)K dz\,, \tag{33}$$

Taking complex conjugate on both sides of equation (18), we get

$$\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* = -W^*\,, \tag{34}$$

Therefore, using (34), we get

$$\int_0^1 W^*\Theta dz = -\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz\,, \tag{35}$$

Taking complex conjugate on both sides of equation (19), we get

$$\left(D^2-a^2-E' p_3\sigma^*\right)\Gamma^* = -W^*\,, \tag{36}$$

Therefore, using (36), we get

$$\int_0^1 W^*\Gamma dz = -\int_0^1 \Gamma\left(D^2-a^2-E' p_3\sigma^*\right)\Gamma^* dz\,, \tag{37}$$

Also taking complex conjugate on both sides of equation (20), we get

$$\left[D^2-a^2-p_2\sigma^*\right]K^* = -DW^*\,, \tag{38}$$

Therefore, equation (38), using appropriate boundary condition (21), we get

$$\int_0^1 W^* D\left(D^2-a^2\right)K dz = -\int_0^1 DW^*\left(D^2-a^2\right)K dz = \int_0^1 K\left(D^2-a^2\right)\left(D^2-a^2-p_2\sigma^*\right)K^* dz\,, \tag{39}$$

Substituting (35), (37) and (39), in the right hand side of equation (33), we get

$$\frac{F}{P_l}\int_0^1 W^*\left(D^2-a^2\right)^2 W dz-\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\right)\int_0^1 W^*(D^2-a^2)W dz = -Ra^2\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz$$

$$+R_s a^2\int_0^1 \Gamma(D^2-a^2-E' p_3\sigma^*)\Gamma^* dz - Q\int_0^1 K\left(D^2-a^2\right)\left(D^2-a^2-p_2\sigma^*\right)K^* dz\,, \tag{40}$$

Integrating the terms on both sides of equation (40) for an appropriate number of times and making use of the appropriate boundary conditions (21), we get

$$\frac{F}{P_l}\int_0^1\left\{\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right\}dz+\left(\frac{\sigma}{\varepsilon}+\frac{1}{P_l}\right)\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz$$

$$=Ra^2\int_0^1\left(\left|D\Theta\right|^2+a^2\left|\Theta\right|^2+Ep_1\sigma^*\left|\Theta\right|^2\right)dz-R_sa^2\int_0^1\left(\left|D\Gamma\right|^2+a^2\left|\Gamma\right|^2+E^/p_3\sigma^*\left|\Gamma\right|^2\right)dz$$

$$-Q\int_0^1\left(\left|D^2K\right|^2+2a^2\left|DK\right|^2+a^4\left|K\right|^2\right)dz-Qp_2\sigma^*\int_0^1\left(\left|DK\right|^2+a^2\left|K\right|^2\right)dz\,, \qquad (41)$$

now equating real and imaginary parts on both sides of equation (41), and cancelling $\sigma_i(\neq 0)$ throughout from imaginary part, we get

$$\frac{F}{P_l}\int_0^1\left\{\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right\}dz+\left(\frac{\sigma_r}{\varepsilon}+\frac{1}{P_l}\right)\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz$$

$$=Ra^2\int_0^1\left(\left|D\Theta\right|^2+a^2\left|\Theta\right|^2+Ep_1\sigma_r\left|\Theta\right|^2\right)dz-R_sa^2\int_0^1\left(\left|D\Gamma\right|^2+a^2\left|\Gamma\right|^2+E^/p_3\sigma_r\left|\Gamma\right|^2\right)dz$$

$$-Q\int_0^1\left(\left|D^2K\right|^2+2a^2\left|DK\right|^2+a^4\left|K\right|^2\right)dz-Qp_2\sigma_r\int_0^1\left(\left|DK\right|^2+a^2\left|K\right|^2\right)dz\,, \qquad (42)$$

and

$$\frac{1}{\varepsilon}\int_0^1\left\{\left|DW\right|^2+a^2\left|W\right|^2\right\}dz=-Ra^2Ep_1\int_0^1\left|\Theta\right|^2dz+R_sa^2E^/p_3\int_0^1\left|\Gamma\right|^2dz+Qp_2\int_0^1\left(\left|DK\right|^2+a^2\left|K\right|^2\right)dz\,, \qquad (43)$$

of which the equation (42) can be rearranged in the form

$$\frac{F}{P_l}\int_0^1\left\{\left|D^2W\right|^2+2a^2\left|DW\right|^2+a^4\left|W\right|^2\right\}dz+\left(\frac{\sigma_r}{\varepsilon}+\frac{1}{P_l}\right)\int_0^1\left(\left|DW\right|^2+a^2\left|W\right|^2\right)dz$$

$$=Ra^2\int_0^1\left(\left|D\Theta\right|^2+a^2\left|\Theta\right|^2\right)dz-R_sa^2\int_0^1\left(\left|D\Gamma\right|^2+a^2\left|\Gamma\right|^2\right)dz-Q\int_0^1\left|\left(D^2-a^2\right)K\right|^2dz$$

$$+\sigma_r\left[Ra^2Ep_1\int_0^1\left|\Theta\right|^2dz-R_sa^2E^/p_3\int_0^1\left|\Gamma\right|^2dz-Qp_2\int_0^1\left(\left|DK\right|^2+a^2\left|K\right|^2\right)dz\right], \qquad (44)$$

The equation (43) together with $\sigma_r\geq 0$, yields the inequality

$$\sigma_r\left[Ra^2Ep_1\int_0^1|\Theta|^2dz - R_s a^2 E^{/} p_3\int_0^1|\Gamma|^2dz - Qp_2\int_0^1\left(|DK|^2 + a^2|K|^2\right)dz\right] \le 0, \tag{45}$$

Now, utilizing the inequality (22), we have

$$\int_0^1\left\{|D\Gamma|^2 + a^2|\Gamma|^2\right\}dz \ge (\pi^2 + a^2)\int_0^1|\Gamma|^2 dz, \tag{46}$$

While from the equation (43), we get

$$\int_0^1|\Gamma|^2dz \ge \frac{1}{R_s a^2 E^{/} p_3\varepsilon}\int_0^1|DW|^2dz - \frac{Qp_2}{R_s a^2 E^{/} p_3\varepsilon}\int_0^1\left(|DK|^2 + a^2|K|^2\right)dz, \tag{47}$$

So that using inequality (47), we can write the inequality (46) as

$$\int_0^1\left\{|D\Gamma|^2 + a^2|\Gamma|^2\right\}dz \ge \frac{(\pi^2 + a^2)}{R_s a^2 E^{/} p_3\varepsilon}\int_0^1|DW|^2dz - \frac{Qp_2(\pi^2 + a^2)}{R_s a^2 E^{/} p_3\varepsilon}\int_0^1\left(|DK|^2 + a^2|K|^2\right)dz, \tag{48}$$

Now, if permissible let $R_s \ge R$, Then in that case we derive from equation (44) and utilizing the inequalities (23), (26), (32), (45) and (48), we get

$$\left[(\pi^2 + a^2)\left\{\frac{F}{P_l} + \frac{1}{E^{/} p_3\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right)\right\} - \frac{R_s a^2}{\pi^2(\pi^2 + a^2)}\right]\int_0^1|DW|^2dz + I_1 \langle 0, \tag{49}$$

Where $I_1 = \left(\frac{a^2F}{P_l} + \frac{\sigma_r}{\varepsilon} + \frac{1}{P_l}\right)\int_0^1\left(|DW|^2 + a^2|W|^2\right)dz + Q\int_0^1\left|\left(D^2 - a^2\right)K\right|^2dz$, is positive definite. Therefore, we must have

$$R_s \rangle \frac{\pi^2(\pi^2 + a^2)^2}{a^2}\left\{\frac{F}{P_l} + \frac{1}{E^{/} p_3\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right)\right\}. \tag{50}$$

and thus we necessarily have

$$R_s \rangle 4\pi^4\left\{\frac{F}{P_l} + \frac{1}{E^{/} p_3\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right)\right\} \tag{51}$$

Since the minimum value of $\frac{\pi^2\left(\pi^2 + a^2\right)^2}{a^2}$ is $4\pi^4$ at $a^2 = \pi^2 \rangle 0$.

Hence, if

$\sigma_r \geq 0$ and $\sigma_i \neq 0$, then $R_s \rangle 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Q p_2}{\pi^2} \right) \right\}$, (52)

And this completes the proof of the theorem.

Presented otherwise from the point of view of existence of instability as stationary convection, the above Theorem 1, can be put in the form as follow:-

Corollary 1: The sufficient condition for the onset of instability as a non-oscillatory motions of non-growing amplitude in a thermosolutal couple-stress viscoelastic fluid configuration of Veronis type in the presence of uniform vertical magnetic field in a porous medium heated from below is that, $R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Q p_2}{\pi^2} \right) \right\}$, where R_s is the Thermosolutal Rayliegh number, Q is the Chandrasekhar number, p_2 is the magnetic Prandtl number, p_3 is the thermosolutal Prandtl number, P_l is the medium permeability, ε is the porosity and F is the couple-stress parameter, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid

or

The onset of instability in a thermosolutal couple-stress viscoelastic fluid configuration of Veronis type in the presence of uniform vertical magnetic field in a porous medium heated from below, cannot manifest itself as oscillatory motions of growing amplitude if the Thermosolutal Rayliegh number R_s, the Chandrasekhar number Q, the magnetic Prandtl number p_2, the thermosolutal Prandtl number p_3, the medium permeability P_l, the porosity ε and the couple-stress parameter F, satisfy the inequality $R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Q p_2}{\pi^2} \right) \right\}$, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid

The sufficient condition for the validity of the 'PES' can be expressed in the form:

Corollary 2: If $(W, \Theta, \Gamma \sigma)$, $\sigma = \sigma_r + i\sigma_i$, $\sigma_r \geq 0$ is a solution of equations (17) – (20), with $R \rangle 0$ and,

$$R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Q p_2}{\pi^2} \right) \right\},$$

Then $\sigma_i = 0$.

In particular, the sufficient condition for the validity of the 'exchange principle' i.e., $\sigma_r = 0 \Rightarrow \sigma_i = 0$ if $R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E^{/} p_3 \varepsilon} \left(1 - \frac{Qp_2}{\pi^2} \right) \right\}$.

In the context of existence of instability in 'oscillatory modes' and that of 'overstability' in the present configuration of Veronis type, we can state the above theorem as follow:-

Corollary 3: The necessary condition for the existence of instability in 'oscillatory modes' and that of 'overstability' in a thermosolutal couple-stress fluid configuration of Veronis type in the presence of uniform vertical magnetic field in a porous medium heated from below is that the Thermosolutal Rayliegh number R_s, the Chandrasekhar number Q, the magnetic Prandtl number p_2, the thermosolutal Prandtl number p_3, the medium permeability P_l, the porosity ε and the couple-stress parameter F must satisfy the inequality $R_s \rangle 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E^{/} p_3 \varepsilon} \left(1 - \frac{Qp_2}{\pi^2} \right) \right\}$, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid.

Special Cases: It follows from theorem 1 that an arbitrary neutral or unstable mode is non-oscillatory in character and 'PES' is valid for:

(i). Thermal convection in couple-stress fluid heated from below, i. e. when Q = 0 = R_s. (Sunil et al [16])

(ii). Magneto-thermal convection in couple-stress fluid heated from below (R_s=0), if

$$\left(\frac{Qp_2}{\pi^2} \right) \leq 1 \text{ ‘}$$

(iii) Thermosolutal convection of Veronis (1965) type in couple-stress fluid heated from below (Q = 0), if

$$R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E^{/} p_3 \varepsilon} \right\}.$$

A similar theorem can be proved for thermosolutal convection in couple-stress fluid configuration of Stern type in a porous medium as follow:

Theorem 2: If R$\langle$ 0, $R_s \langle 0$, F$\rangle$ 0, $P_l \rangle 0$, $p_1 \rangle 0, p_3 \rangle 0, \sigma_r \geq 0$, $\sigma_i \neq 0$, $\frac{Qp_2}{\pi^2} \leq 1$ and $|R| \geq |R_s|$ then the necessary condition for the existence of non-trivial solution (W,Θ,Γ) of equations (17) – (20), together with boundary conditions (21) is that

$$|R|\rangle 4\pi^4\left\{\frac{F}{P_l}+\frac{1}{Ep_1\varepsilon}\left(1-\frac{Qp_2}{\pi^2}\right)\right\}.$$

Proof: Replacing R and R_s by $-|R|$ and $-|R_s|$, respectively in equations (17) – (20) and proceeding exactly as in Theorem 1 and utilizing the inequality (29), we get the desired result.

Presented otherwise from the point of view of existence of instability as stationary convection, the above Theorem 2, can be put in the form as follow:-

Corollary 4: The sufficient condition for the onset of instability as a non-oscillatory motions of non-growing amplitude in a thermosolutal couple-stress fluid configuration of Stern type in the presence of uniform vertical magnetic field in a porous medium is that, $|R|\le 4\pi^4\left\{\frac{F}{P_l}+\frac{1}{Ep_1\varepsilon}\left(1-\frac{Qp_2}{\pi^2}\right)\right\}$, where R is the Thermal Rayliegh number, Q is the Chandrasekhar number, p_2 is the magnetic Prandtl number, p_1 is the thermal Prandtl number, P_l is the medium permeability, ε is the porosity and F is the couple-stress parameter, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid

or

The onset of instability in a thermosolutal couple-stress fluid configuration of Stern type in the presence of uniform vertical magnetic field in a porous medium, cannot manifest itself as oscillatory motions of growing amplitude if the Thermal Rayliegh number R, the Chandrasekhar number Q, the magnetic Prandtl number p_2, the thermal Prandtl number p_1, the medium permeability P_l, the porosity ε and the couple-stress parameter F, satisfy the inequality $|R|\le 4\pi^4\left\{\frac{F}{P_l}+\frac{1}{Ep_1\varepsilon}\left(1-\frac{Qp_2}{\pi^2}\right)\right\}$, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid

The sufficient condition for the validity of the 'PES' can be expressed in the form:

Corollary 5: If $(W,\Theta,\Gamma\sigma)$, $\sigma=\sigma_r+i\sigma_i$, $\sigma_r\ge 0$ is a solution of equations (17) – (20), with R$\rangle$ 0 and,

$$|R| \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{Ep_1\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right) \right\} ,$$

Then $\sigma_i = 0$.

In particular, the sufficient condition for the validity of the 'exchange principle' i.e., $\sigma_r = 0 \Rightarrow \sigma_i = 0$ if $|R| \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{Ep_1\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right) \right\}$.

In the context of existence of instability in 'oscillatory modes' and that of 'overstability' in the present configuration of Stern's type, we can state the above theorem as follow:-

Corollary 6: The necessary condition for the existence of instability in 'oscillatory modes' and that of 'overstability' in a thermosolutal couple-stress fluid configuration of Stern type in the presence of uniform vertical magnetic field in a porous medium is that the Thermal Rayliegh number R, the Chandrasekhar number Q, the magnetic Prandtl number p_2, the thermal Prandtl number p_1, the medium permeability P_l, the porosity ε and the couple-stress parameter F must satisfy the inequality $|R| \rangle 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{Ep_1\varepsilon}\left(1 - \frac{Qp_2}{\pi^2}\right) \right\}$, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid.

Special Cases: It follows from theorem 1 that an arbitrary neutral or unstable mode is non-oscillatory in character and 'PES' is valid for:

(i). Thermal convection in couple-stress fluid i. e. when Q = 0 =R.

(ii). Magneto-thermal convection couple-stress fluid (R=0), if

$$\left(\frac{Qp_2}{\pi^2} \right) \leq 1 .$$

(iii). Thermosolutal convection of Stren [23] type in couple-stress fluid (Q = 0), if

$$|R| \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{Ep_1\varepsilon} \right\}.$$

5. CONCLUSIONS

Theorem 1 mathematically established that the onset of instability in a thermosolutal couple-stress fluid configuration of Veronis type in the presence of uniform vertical magnetic field in a porous medium, cannot manifest itself as oscillatory motions of growing amplitude if the Thermosolutal Rayliegh number R_s, the Chandrasekhar number

Q, the magnetic Prandtl number p_2, the thermosolutal Prandtl number p_3, the medium permeability P_l, the porosity ε and the couple-stress parameter F satisfy the inequality $R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Qp_2}{\pi^2} \right) \right\}$, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid

The essential content of the theorem 1, from the point of view of linear stability theory is that for the thermosolutal configuration of Veronis type of couple-stress fluid of infinite horizontal extension in the presence of uniform vertical magnetic field in a porous medium, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid, an arbitrary neutral or unstable modes of the system are definitely non-oscillatory in character if $R_s \leq 4\pi^4 \left\{ \frac{F}{P_l} + \frac{1}{E' p_3 \varepsilon} \left(1 - \frac{Qp_2}{\pi^2} \right) \right\}$, and in particular PES is valid.

The similar conclusions can be drawn for the thermosolutal configuration of Stern type of couple-stress fluid of infinite horizontal extension in the presence of uniform vertical magnetic field in a porous medium, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid from Theorem 2.

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On the Bounds for Oscillations in Double Diffusive Convection with Cross-Diffusions Effects and Variable Viscosity

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Abstract: In the present paper, the stability analysis of double-diffusive convection problems (Veronis and Stern Types) with cross-diffusions effects (Soret and Dufour effects) have been carried out mathematically with temperature dependent (variable)viscosity. The eigenvalues equations governing these problems have been casted into mathematically tractable forms for stability analysis using some linear transformations. The stability of the oscillatory modes and consequently the bounds for the complex growth rate for arbitrary neutral or unstable oscillatory perturbations are derived which are valid for each combinations of rigid (slip free) and dynamically free (stress-free) boundaries and are of general nature. Various consequences of the derived results are also worked out.

Keywords: Double-Diffusive Convection; oscillatory motions; complex growth rate; temperatre-dependent viscosity; eigenvalue problem; Soret effect; Dufour effect.

1. INTRODUCTION

A broader range of dynamical behaviour is observed in the convective motions that may occur in a gravitational field containing two components (for example, temperature and solute) of different diffusivities that affect the density of the fluid and the phenomenon is known as double-diffusive convection. The phenomena of double-diffusive convection occur when the temperature and concentration gradients are of comparable magnitude and operate on different scales and lead to large scale convection. These kinds of double-diffusion processes are found in astrophysics (big Helium-stars), the earth core, metal alloy, refilling of gas reservoirs, etc. Double-diffusive convection is also of importance in various other fields of practical interest such as high quality crystal production oceanography, production of pure medication, solidification of molten alloys, limnology and engineering.

The double diffusive process was first recognized by Stommel et. al. [1] through his '*thought experiment*' with ocean flow/circulation in mind. Two fundamental configurations have been studied in the context of thermohaline instability problems, the first one by Stern [2], wherein the temperature gradient is stabilizing and the concentration gradient is destabilizing and the second one by Veronis [3], wherein the temperature gradient is destabilizing and the concentration gradient is stabilizing. Stern found that the steady motion is the preferred mode of onset of instability whereas Veronis observed that oscillatory mode of instability is the preferred mode of convective instability. Since then numerous authors have investigated the double diffusive convection problems under varying assumptions of hydrodynamics both numerically and analytically. For a broader view of the subject of double-diffusive convection one may refer to Turner [4], Brandt and Fernando [5], Schmitt [6] and Nield [7].

The stability properties of binary fluids are quite different from pure fluids because of Soret and Dufour effects. An externally imposed temperature gradient produces a chemical potential gradient and the phenomena known as the Soret effect, whereas the analogous effect that arises from a concentration gradient which produces a heat flux is called the Dufour effect. The stability of Dufour-Soret driven double-diffusive convection in a horizontal layer of a fluid subjected to thermal and solutal gradients has been investigated theoretically by means of a linear stability analysis by many authors including Groot and Mazur [8], Fitts [9] and McDougall [10]. It is well known fact that the viscosity is one of the properties of a fluid which are most sensitive to temperature and the variation of viscosity of liquids with temperature is extremely rapid (*cf.* Straughan [11]) which plays an important role in several physical situations wherein the fluid viscosity is a function of temperature and/or depth.

Dhiman and Kumar [12] have investigated the stability of oscillatory modes for thermohaline configuration with temperature dependent viscosity and derived a condition for the stability of oscillatory modes and obtained the bounds for complex growth rate of arbitrary neutral or unstable perturbations. These results have been recently improved upon by Dhiman.et. al. [13] by eliminating the curious condition on $D^2 f(\geq 0)$, where, f is the temperature dependent viscosity function and D^2 represents the double derivative with respect to z.

Motivated by the above analysis and discussions, the aim of the present paper is to extend the analysis of Dhiman et. al. [13] to a more general problem, namely Double-Diffusive Convection with Cross-Diffusions, when viscosity of the fluid is temperature dependent. Here, we shall investigate the stability of the oscillatory motions and derive the bounds for complex growth rate, if they exists. In the present analysis, some non-trivial integral estimates obtained from the governing eigenvalue equations are used to obtain these results, which are also free from the curious condition; $D^2 f \geq 0$.The present analysis is thus an attempt to study the effects of viscosity variation and cross diffusions on the onset of double diffusive convection for general cases of boundary conditions.

2. PHYSICAL CONFIGURATION AND EIGEN VALUE PROBLEM

Consider a viscous, incompressible (Boussinesq) fluid of infinite horizontal extension and finite vertical depth statically confined between two horizontal boundaries $z = 0$ and $z = d$ at constant temperatures T_0 and $T_1 (T_0 > T_1)$ at the lower and upper boundaries respectively,and uniform concentrations S_0 and S_1 $(S_0 > S_1)$, in the force field of the gravity. The uniform temperature gradient $\left(\beta = \frac{T_0 - T_1}{d}\right)$and concentration gradient $(\beta' = \frac{C_0 - C_1}{d})$ make opposing contributions to the vertical density $\rho = \rho_0[1 + \alpha\beta z + \alpha'\beta' z]$, where, α and α' are respectively the coefficients of thermal expansion and analogous concentration expansion. The extra effects that we have considered here are that of coupled fluxes of the two properties due to irreversible thermodynamic effects; namely Soret and Dufour effects.

Following the usual steps of linear stability theory, the non-dimensional linearized perturbation equations and the boundary conditions governing the onset of Double-Diffusive Convection in the presence of Soret and Dufour effects with temperature dependent (variable) viscosity are given by (*cf.* Dhiman and Kumar [12]);

$$f(D^2 - a^2)^2 w - \frac{\mathrm{p}}{\sigma}(D^2 - a^2)w + 2(Df)D(D^2 - a^2)w + D^2 f(D^2 + a^2)w =$$

$$= R_T a^2 \theta - R_s a^2 \varphi \tag{1}$$

$$(D^2 - a^2 - \mathrm{p})\theta + D_T(D^2 - a^2)\varphi = -w \tag{2}$$

$$\left(D^2 - a^2 - \frac{\mathrm{p}}{\tau}\right)\varphi + S_T(D^2 - a^2)\theta = -\frac{w}{\tau} \tag{3}$$

The above equations must be solved subject to either of the boundary conditions;

$$w = 0 = \theta = \varphi = D^2 w \quad \text{at } z = 0 \text{ and } z = 1 \tag{4}$$

(Both boundaries dynamically free)

$$w = 0 = \theta = \varphi = Dw \text{ at } z = 0 \text{ and } z = 1 \tag{5}$$

(Both boundaries rigid)

$$w = 0 = \theta = \varphi = Dw \text{ at } z = 0 \text{ and } w = 0 = \theta = \varphi = D^2 w \text{ at } z = 1 \tag{6}$$

(Lower boundary rigid and upper boundary dynamical free)

$$w = 0 = \theta = \varphi = D^2 w \text{ at } z = 0 \text{ and } w = 0 = \theta = \varphi = Dw \text{ at } z = 1 \tag{7}$$

(Lower boundary dynamical free and upper boundary rigid)

The system of equations (1)-(3) together with either of the boundary conditions (4)-(7) thus constitutes an *eigenvalue problem* for p for given values of other parameters; namely $a^2, \sigma, R_T, R_S, \tau, S_T$ *and* D_T. Further, a given state of the system is *stable*, *neutral* or *unstable* according as p_r (real part of p) is negative, zero or positive respectively. Further, if $p_r = 0$ implies $p_i = 0$ for every wave number a, then the *principle of exchange of stabilities* (PES) is valid, which means that instability sets in as stationary convection, otherwise we shall have overstability at least when instability sets in as certain modes.

Further, we note that the mathematical structure of the system of equations (1)-(3) governing Double-Diffusive Convection in the presence of coupled Soret and Dufour effects with temperature dependent viscosity is qualitatively different from those governing double diffusive convection problems in the absence of these effects, since the latter involves the coupling amongst the eigen-functions $w, \theta, and\ \varphi$ and thus obstructs any attempt for the elegant extension of the results derived in double diffusive convection problems. The nasty behaviour of these equations is arrested by introducing some *indigenous* linear transformations.

Let us introduce the transformations;

$$w = \frac{\tau w'}{S_T - K}; \theta = \frac{K\theta'}{(KE + S_T F\tau)} + \frac{\tau F\varphi'}{(KE + S_T F\tau)}; \text{ and} \varphi = \frac{\tau S_T}{(KE + S_T F\tau)}\theta' - \frac{E\tau}{(KE + S_T F\tau)}\varphi'$$

where,

$$B = -\frac{K}{\tau}, E = \left(\frac{S_T + B}{D_T + K}\right)K,\ F = \left(\frac{S_T + B}{D_T + K}\right)D_T$$

and K is any positive root of the equation $K^2 + K(\tau - 1) - \tau S_T D_T = 0.$

Now, using the above transformations in equations (1)-(3) and in boundary conditions (4)-(7) and dropping the dashes for convenience in writing, we have the following reduced forms of equations

$$f(D^2 - a^2)^2 w - \frac{p}{\sigma}(D^2 - a^2)w + 2(Df)D(D^2 - a^2)w + D^2 f(D^2 + a^2)w = R'_T a^2\theta - R'_S a^2\varphi \quad (8)$$

$$[K_1(D^2 - a^2) - p]\theta = -w \quad (9)$$

$$\left[K_2(D^2 - a^2) - \frac{p}{\tau}\right]\varphi = -\frac{w}{\tau} \quad (10)$$

together with either of the boundary conditions (4)-(7).

where, $R'_T = \frac{(D_T + K)(R_T B + R_S S_T)}{BK - S_T D_T}$, $R'_S = \frac{(S_T + B)(R_S K + R_T D_T)}{BK - S_T D_T}$ are respectively the effective thermal and Salinity Rayleigh numbers and $K_1 = 1 + \frac{\tau S_T D_T}{K}$, $K_2 = 1 - \frac{S_T D_T}{K}$ are non-negative constants since $\frac{S_T D_T}{K} \geq 0$ as $S_T, D_T > 0$ and K is also positive constants, as defined earlier.

Remark 1: The above system of equations (8)-(10) and boundary conditions (4)-(7) governing the eigenvalue problem of the present problem yields eigenvalue problem governing;

- **i)** **Double Diffusive Convection (DDC)** with variable viscosity, if we take $D_T = S_T = 0$.Consequently,$K_1 = 1 = K_2$ and $R_T{}' = R_T$(the usual thermal Rayleigh number), and $R_S{}' = R_S$(the usual solutal Rayleigh number).
- **ii)** **Soret Driven Double-Diffusive Convection (SDDDC)** with variable viscosity**,** if we take $D_T = 0$. Consequently, $K_1 = K_2 = 1$ and $R_T{}' = R_T - \frac{\tau R_S S_T}{1-\tau}$ is the modified thermal thermal Rayleigh number, and $R_S{}' = R_T - \frac{\tau R_S S_T}{1-\tau}$ is the modified solutal Rayleigh number.
- **iii)** **Dufour Driven Double-Diffusive Convection (DDDDC)** with variable viscosity**,** if we take $S_T = 0$. Consequently, $K_1 = K_2 = 1$ and $R_T{}' = R_T + \frac{R_T D_T}{1-\tau}$ is the modified thermal Rayleigh number, and $R_S{}' = R_T + \frac{\tau R_T D_T}{1-\tau}$ is the modified solutal Rayleigh number.
- **iv)** Further, when $f = 1$, the above eigenvalue problems refer to the respective configurations with constant viscosity.
- **v)** The system of equations (8)-(10) together with boundary conditions (4)-(7) describes the **Veronis Type Configuration**, when $R'_T > 0 \; and \; R'_S > 0$, whereas it describes the **Stern Type Configuration**, when $R'_T < 0 \; and \; R'_S < 0$.

3. MATHEMATICAL ANALYSIS

Stability of The Oscillatory Modes

In the following theorem, we shall investigate the stability of the oscillatory modes for Veronis type configuration;

Theorem 1. If $(p, w, \theta, \varphi), p = p_r + ip_i,\; p_i \neq 0,$ is a non-trivial solution of equations (8)-(10) together with one of the boundary conditions (4)-(7), $R_T{}' > 0\,, R_S{}' > 0 \; and \;\; R_T{}' \leq \frac{27\pi^4 K_1}{4}\left[f_{min} + \frac{\tau K_2}{\sigma}\right]$, then $p_r < 0$**.**

Proof: Multiplying both sides of the equation (8) by w^* and integrating the resulting equation over the range of z, we get

$$\int_0^1 w^* \left[f(D^2 - a^2)^2 w - \frac{p}{\sigma}(D^2 - a^2)w + 2(Df)D(D^2 - a^2)w + D^2 f(D^2 + a^2)w \right] dz = R_T{}' a^2 \int_0^1 \theta w^* dz - R_S{}' a^2 \int_0^1 \varphi w^* \, dz \qquad (11)$$

Taking complex conjugate of both sides of equations (9) and (10) and using the resulting equations respectively in the first two terms in the right hand side of equation (11), we get

$$\int_0^1 w^*\left[f(D^2-a^2)^2w-\frac{p}{\sigma}(D^2-a^2)w+2(Df)D(D^2-a^2)w+D^2f(D^2+a^2)w\right]dz = -R_T{}'a^2\int_0^1\theta[K_1(D^2-a^2)-p^*]\theta^*dz+R_S{}'a^2\int_0^1\varphi\left[K_2(D^2-a^2)-\frac{p^*}{\tau}\right]\varphi^*dz \qquad (12)$$

Now, integrating the various terms of equation (12) by parts an appropriate number of times and using the relevant boundary conditions (4)-(7), we have

$$\int_0^1 f[|D^2w|^2+a^4|w|^2+2a^2|Dw|^2]dz+\frac{p}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)dz+a^2\int_0^1(|Dw|^2+a^2|w|^2)dz+a^2\int_0^1 D^2f|w|^2dz =$$
$$R_T{}'a^2\int_0^1 K_1[|D\theta|^2+a^2|\theta|^2]dz-R_s{}'a^2\int_0^1\tau K_2[|D\varphi|^2+a^2|\varphi|^2]dz$$
$$+a^2p^*\left[R_T{}'\int_0^1|\theta|^2dz-R_S{}'\int_0^1|\varphi|^2dz\right]=0 \qquad (13)$$

Equating the real and imaginary parts of equation (13) to zero and cancelling $p_i(\neq 0)$ throughout from the imaginary part, we get

$$\int_0^1 f[|D^2w|^2+a^4|w|^2+2a^2|Dw|^2]\,dz+$$
$$\frac{p_r}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)dz+a^2\int_0^1 D^2f|w|^2dz-R_T{}'a^2K_1\int_0^1[|D\theta|^2+a^2|\theta|^2]\,dz+$$
$$R_S{}'a^2\tau K_2\int_0^1(|D\varphi|^2+a^2|\varphi|^2)dz-a^2p_r\left[R_T{}'\int_0^1|\theta|^2dz-R_S{}'\int_0^1|\varphi|^2\,dz\right]=0 \qquad (14)$$

and

$$\frac{1}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)dz+R_T{}'a^2\int_0^1|\theta|^2dz-R_s{}'a^2\int_0^1|\varphi|^2\,dz=0 \qquad (15)$$

If permissible, let $p_r \geq 0$

Now, multiplying equation (15) by p_r and adding the resulting equation to equation (14), we obtain

$$\int_0^1 f[|D^2w|^2+a^4|w|^2+2a^2|Dw|^2]\,dz+$$
$$\frac{2p_r}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)dz+a^2\int_0^1 D^2f|w|^2dz-R_T{}'a^2K_1\int_0^1[|D\theta|^2+a^2|\theta|^2]\,dz+$$
$$R_S{}'a^2\tau K_2\int_0^1\int_0^1[|D\varphi|^2+a^2|\varphi|^2]\,dz=0 \qquad (16)$$

Equation (15) implies that

$$\frac{1}{\sigma}\int_0^1(|Dw|^2+a^2|w|^2)dz<R_s{}'a^2\int_0^1|\varphi|^2\,dz \qquad (17)$$

Since, w, θ and φ vanish at $z=0$ and $z=1$, therefore Rayleigh Ritz inequality (Schultz [14]) yields

$$\int_0^1|Dw|^2dz\geq\pi^2\int_0^1|w|^2\,dz \qquad (18)$$

$$\int_0^1|D\theta|^2dz\geq\pi^2\int_0^1|\theta|^2\,dz \qquad (19)$$

$$\int_0^1 |D\varphi|^2\,dz \geq \pi^2 \int_0^1 |\varphi|^2\,dz \tag{20}$$

Now, combining inequalities (17) and (18), we have

$$\frac{(\pi^2+a^2)}{\sigma}\int_0^1 |w|^2\,dz < R_s{}' a^2 \int_0^1 |\varphi|^2\,dz \tag{21}$$

Also, upon using inequality (20), we can have

$$R_S{}' a^2 \int_0^1 [|D\varphi|^2 + a^2|\varphi|^2]\,dz \geq \frac{(\pi^2+a^2)^2}{\sigma}\int_0^1 |w|^2\,dz \tag{22}$$

Now, utilizing Schwartz inequality, we have

$$\pi^2 \int_0^1 |w|^2 dz \leq \int_0^1 |Dw|^2 dz \leq \left| -\int_0^1 w^* D^2 w \right| dz \leq \int_0^1 [|w|^2 dz]^{\frac{1}{2}} \int_0^1 [|D^2 w|^2 dz]^{\frac{1}{2}}$$

which on simplification yields

$$\int_0^1 [|D^2 w|]^2 dz \geq \pi^4 \int_0^1 |w|^2\,dz \tag{23}$$

Using inequalities (18) and (23), we have

$$\int_0^1 f[|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]\,dz \geq f_{min}(\pi^2 + a^2)^2 \int_0^1 |w|^2 dz \tag{24}$$

where, $f_{min.}$ is the minimum value of f in the closed interval$[0,1]$.

Now, multiplying equation (9) by its complex conjugate and integrating the various terms on left hand side of the resulting equation by parts an appropriate number of times and making use of relevant boundary conditions; $\theta(0) = \theta(1) = 0$, we obtain

$$\int_0^1 {K_1}^2\, |(D^2 - a^2)\theta|^2 dz + 2p_r K_1 \int_0^1 [|D\theta|^2 + a^2|\theta|^2]\,dz + |p|^2 \int_0^1 |\theta|^2 dz = \int_0^1 |w|^2\,dz \tag{25}$$

Since, $p_r \geq 0$, therefore equation (25) gives

$${K_1}^2 \int_0^1 |(D^2 - a^2)\theta|^2 dz \leq \int_0^1 |w|^2 dz \tag{26}$$

Further, emulating the derivation of inequalities (23) and (24), we have the following inequality

$$\int_0^1 |(D^2 - a^2)\theta|^2 = \int_0^1 [|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2 dz] \geq (\pi^2 + a^2)^2 \int_0^1 |\theta|^2\,dz \tag{27}$$

Now, combining inequalities (26) and (27)

$$\int_0^1 |w|^2 dz \ \geq (\pi^2 + a^2)^2 {K_1}^2 \int_0^1 |\theta|^2\,dz \tag{28}$$

Again, we know that

$\int_0^1 |w|^2 dz = \int_0^1 [|w|^2 dz]^{\frac{1}{2}} \int_0^1 [|w|^2 dz]^{\frac{1}{2}}$ (29)

which upon using inequalities (26)and (28), we have

$$\int_0^1 |w|^2\, dz \geq (\pi^2 + a^2){K_1}^2 \left\{\int_0^1 |(D^2 - a^2)\theta|^2 dz\right\}^{\frac{1}{2}} \left\{\int_0^1 |\theta|^2 dz\right\}^{\frac{1}{2}}$$

$\geq (\pi^2 + a^2){K_1}^2 \left| -\int_0^1 \theta^*(D^2 - a^2)\theta \right|$ (using Schwartz inequality)

$\geq (\pi^2 + a^2){K_1}^2 \int_0^1 [|D\theta|^2 + a^2|\theta|^2]$ (30)

Let us consider the integral

$\int_0^1 (fw^*)D^2 w dz = -\int_0^1 (fDw^* + Dfw^*)\, Dwdz$

$= -\int_0^1 f|Dw|^2\, dz - \int_0^1 w^* DfDwdz$ (31)

Let, $I = \int_0^1 w^* DfDwdz = -\int_0^1 (w^* D^2 f + DfDw^*) wdz = -\int_0^1 (D^2 f)|w|^2 dz - I^*$(32)

which implies that

$I + I^* = 2Re\ (I) = -\int_0^1 D^2 f |w|^2 dz$ (33)

Where Re stands for real part of the quantity.

Also, from inequalities (31) and (33), we have

$\int_0^1 D^2 f|w|^2 dz = 2Re \int_0^1 (fw^*)D^2 wdz + \int_0^1 f|Dw|^2 dz$ (34)

Also, $\int_0^1 f|(D^2 + a^2)w|^2\, dz = \int_0^1 f(D^2 w + a^2 w)(D^2 w^* + a^2 w^*) dz$

$$= \int_0^1 [|D^2 w|^2 + a^4 |w|^2]\, dz + 2a^2 Re\left(\int_0^1 (fw^*)D^2 wdz\right)$$

which yields

$\int_0^1 [|D^2 w|^2 + a^4|w|^2]\, dz = \int_0^1 f|(D^2 + a^2)w|^2\, dz - 2a^2 Re\left(\int_0^1 (fw^*)D^2 wdz\right)$ (35)

Further, in view of equations (34) and (35), we can have

$$\int_0^1 f[|D^2 w|^2 + 2a^2|Dw|^2 + a^4|w|^2]dz + a^2 \int_0^1 (D^2 f)|w|^2\, dz$$

$$= \int_0^1 f|(D^2 + a^2)w|^2\, dz + 4a^2 \int_0^1 f|Dw|^2 dz \geq f_{min.} \int_0^1 (\pi^4 + a^4 + 2a^2\pi^2)|w|^2 dz$$

$\geq f_{min.}(\pi^2 + a^2)^2 \int_0^1 |w|^2 \, dz$ (36)

Using inequalities (22), (30), (36), in equation (16) and the fact that $p_r \geq 0$, we have

$\frac{K_1(\pi^2+a^2)^3}{a^2}\left(f_{min.} + \frac{\tau K_2}{\sigma}\right)\int_0^1 |w|^2 \, dz < R_T{}' \int_0^1 |w|^2 \, dz$ (37)

Since, the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ with respect to a^2 is $\frac{27\pi^4}{4}$, therefore inequality (37) gives

$$\left[K_1 \frac{27\pi^4}{4}\left(f_{min.} + \frac{\tau K_2}{\sigma}\right) - R_T{}'\right]\int_0^1 |w|^2 dz < 0$$

The above inequality clearly implies that

$\frac{27\pi^4 K_1(\tau K_2 + \sigma f_{min.})}{4 R_T{}' \sigma} < 1$ (38)

which is a contradiction to the hypothesis of the theorem.

Hence, we must have $p_r < 0$.

This completes the proof of the theorem.

The above theorem clearly implies that the oscillatory modes of system are stable, when $R_T{}' \leq \frac{27\pi^4 K_1}{4}\left(f_{min} + \frac{\tau K_2}{\sigma}\right)$. Alternatively, one can also say that the oscillatory modes of growing amplitude are not allowed in Double-Diffusive convection problem (Veronis type) in the presence of coupled effects and with variable viscosity, if $R_T{}' \leq \frac{27 \;\; ^4 K_1}{4}\left(f_{min} + \frac{\tau K_2}{\sigma}\right)$.

It is to note that this sufficient condition for the stability of the oscillatory modes is independent of the condition; $D^2 f \geq 0$ on the double derivative of the temperature dependent viscosity function (*cf.* Dhiman et. al [14]).

Further, in view of Remark 1 above, we have the following corollaries;

Corollary 1: Under the hypothesis of Theorem 1, for DDC with variable viscosity, if $R_T \leq \left(f_{min} + \frac{\tau}{\sigma}\right)$, then $p_r < 0$.

Corollary 2: Under the hypothesis of Theorem 1, for SDDDC with variable viscosity, $R_T \leq \left(f_{min} + \frac{\tau}{\sigma} + \frac{\tau R_S S_T}{1-\tau}\right)$, then $p_r < 0$.

Corollary 3: Under the hypothesis of Theorem 1, for DDDDC with variable viscosity, $R_T \leq \left(f_{min} + \frac{\tau}{\sigma} - \frac{R_T D_T}{1-\tau}\right)$,then $p_r < 0$.

It is to note that when the viscosity is constant or varying linearly or exponentially, we have $f_{min} = 1$, and Corollary 1 implies that for DDC problem if $R_T \leq \left(1 + \frac{\tau}{\sigma}\right)$, then $p_r < 0$, a result obtained by Gupta et.al. [15].

We shall now derive an analogous result for Stern's type configuration.

Theorem 2. If $(p, w, \theta, \varphi), p = p_r + ip_i, p_i \neq 0$ is a nontrivial solution of equation (8)-(10) together with one of the boundary conditions (4)-(7) and $R_T{}' < 0, R_S{}' < 0$ and $|R_S{}'| \leq \frac{27\ {}^4K_1}{4}\left[f_{min} + \frac{\tau K_2}{\sigma}\right]$, then $p_r < 0$.

Proof. Following the analysis adopted in the derivation of the result for the case of Veronis type configuration, analogous result can be easily derived for the case of Stern's type Double-Diffusive Convection in the presence of coupled effects with temperature dependent viscosity, just by replacing $R_T{}'$and $R_S{}'$ with$-|R_T{}'|$and $-|R_S{}'|$respectively in Theorem 1.

Further, we can easily obtain the analogous results contained in Corollaries 1-3 Stern type configuration.

In the following analysis, we have derived bounds which arrest the complex growth rate of the arbitrary neutral or unstable $(p_r \geq 0)$ oscillatory motions $(p_i \neq 0)$.

Bounds for the Complex Growth Rate

Theorem 3: If (p, w, θ, φ) $p = p_r + ip_i, p_r \geq 0, p_i \neq 0$ is a non-trivial solution of equations (8)-(10) together with one of the boundary conditions (4)-(7), $R_T{}' > 0$, $R_S{}' > 0$, then

$|p| < \frac{R_T{}'\sigma\sqrt{M^2-1}}{(\sigma f_{min.} + \tau K_2)}$, where, $M = \frac{4R_T{}'\sigma}{27\pi^4 K_1(\tau K_2 + \sigma f_{min.})}$.

Proof: Proceeding exactly as in Theorem1, utilizing the fact that $p_r \geq 0$, we have from equation (25), the following inequality

$$K_1{}^2 \int_0^1 |(D^2 - a^2)\theta|^2\, dz + |p|^2 \int_0^1 |\theta|^2 dz < \int_0^1 |w|^2\, dz \tag{39}$$

Using inequality (27), inequality (39) gives

$$K_1{}^2(\pi^2 + a^2)^2 \left[1 + \frac{|p|^2}{K_1{}^2(\pi^2 + a^2)^2}\right] \int_0^1 |\theta|^2\, dz \leq \int_0^1 |w|^2\, dz \tag{40}$$

Now,

$$\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz = \left| - \int_0^1 \theta^*(D^2 - a^2)\theta dz \right| \leq \left| \int_0^1 \theta(D^2 - a^2)\theta dz \right|$$

$$\leq \left|\int_0^1 |\theta||(\mathrm{D}^2 - a^2)\theta| dz\right| \leq \left[\int_0^1 |\theta|^2\right]^{\frac{1}{2}} \left[\int_0^1 |(D^2 - a^2)|^2 dz\right]^{\frac{1}{2}} \quad \text{(using Schwartz inequality)}$$

which upon using inequality (40) yields

$$\int_0^1 [|D\theta|^2 + a^2|\theta|^2] dz < \frac{1}{{K_1}^2(\pi^2+a^2)^2} \left[1 + \frac{|p|^2}{{K_1}^2(\pi^2+a^2)^2}\right]^{\frac{-1}{2}} \int_0^1 |w|^2 dz \tag{41}$$

Now, making use of inequalities (22), (37) and (41) and using the fact that $p_r \geq 0$, equation (16) implies that

$$\frac{(\pi^2+a^2)^3}{a^2} \left[\left(f_{min.}K_1 + \frac{\tau K_1 K_2}{\sigma}\right) - \frac{{R_T}'}{\left[1 + \frac{|p|^2}{{K_1}^2(\pi^2+a^2)^2}\right]^{\frac{1}{2}}}\right] \int_0^1 |w|^2\, dz < 0 \tag{42}$$

Utilizing the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ with respect to a^2 as $\frac{27\ {}^{4}}{4}$ in inequality (42), we get

$$\left[1 + \frac{|p|^2}{{K_1}^2(\pi^2+a^2)^2}\right]^{\frac{1}{2}} < \frac{4{R_T}'\sigma}{K_1 27\pi^4(\sigma f_{min.} + \tau K_2)}$$

which can be written as

$$|p| < K_1(\pi^2 + a^2)\sqrt{M^2 - 1} \tag{43}$$

where, $M = \frac{4{R_T}'\sigma}{27\pi^4 K_1(\tau K_2 + \sigma f_{min.})}$.

Since, $\left[1 + \frac{|p|^2}{{K_1}^2(\pi^2+a^2)^2}\right]^{\frac{1}{2}} \geq 0$, therefore it follows from inequality (42) that

$$(\pi^2 + a^2) < \frac{{R_T}'a^2\sigma}{K_1(\pi^2+a^2)^2(\sigma f_{min.} + \tau K_2)} \tag{44}$$

which upon using the minimum value of $\frac{(\pi^2+a^2)^2}{a^2}$ with respect to a^2 is $4\pi^2$, yields

$$(\pi^2 + a^2) < \frac{{R_T}'\sigma}{K_1(\sigma f_{min.} + \tau K_2)4\pi^2} \tag{45}$$

Using inequality (45) in inequality (43), we obtained

$$|p| < \frac{{R_T}'\sigma}{(\sigma f_{min.} + \tau K_2)4\pi^2}\sqrt{M^2 - 1}\,. \tag{46}$$

This completes the proof of the theorem.

From the point of view of hydrodynamic theory, we may state the above theorem as;

The complex growth rate $p = p_r + ip_i$ of an arbitrary oscillatory perturbation of growing amplitude ($p_r \geq 0$) lies inside a semi-circle in the right half of the $\mathrm{p_r p_i}$ − plane whose centre is at the origin and whose radius is given by $|p| < \frac{R_T{}'\sigma}{(\sigma f_{min.}+\tau K_2)4\pi^2}\sqrt{M^2-1}.$

Further, in view of Remark 1 above, we have the following corollaries;

Corollary 4: Under the hypothesis of Theorem 3, for DDC with variable viscosity, $|p| < \frac{R_T\sigma}{(\sigma f_{min}+\tau)4\pi^2}\sqrt{M'^2-1}$, where, $M' = \frac{4\sigma\lambda R_S}{27\ \ ^4(\tau+\sigma f_{min})}$.

Corollary 5: Under the hypothesis of Theorem 3, for SDDDC with variable viscosity, $|p| < \frac{\left(R_T-\frac{\tau R_S S_T}{1-\tau}\right)\sigma}{(\sigma f_{min}+\tau)4\pi^2}\sqrt{M''^2-1}$, where, $M'' = \frac{4\sigma\left(R_T-\frac{\tau R_S S_T}{1-\tau}\right)}{27\ \ ^4K_1(\tau K_2+\sigma f_{min})}$.

Corollary 6: Under the hypothesis of Theorem 3, for DDDDC with variable viscosity, $|p| < \frac{\left(R_T+\frac{R_T D_T}{1-\tau}\right)\sigma}{(\sigma f_{min}+\tau)4\pi^2}\sqrt{M'''^2-1}$, where, $M''' = \frac{4\sigma\left(R_T+\frac{R_T D_T}{1-\tau}\right)}{27\ \ ^4K_1(\tau K_2+\sigma f_{min})}$.

It is to note that when the viscosity is constant or varying linearly or exponentially, we have $f_{min} = 1$, and Corollary 4 yields the bound for DDC problem as derived by Gupta et.al. [15].

We shall now derive the analogous bound for Stern's type configuration.

Theorem 4. If $(p, w, \theta, \varphi), p = p_r + ip_i, p_i \neq 0$ is a non-trivial solution of equations (8)-(10) together with one of the boundary conditions (4)-(7) and $R_T{}' < 0, R_S{}' < 0$, then

$|p| < \frac{|R_S{}'|\sigma\sqrt{N'^2-1}}{4\pi^2(K_1+\sigma f_{min})}$, where, $N' = \frac{4|R_S{}'|\sigma}{27\pi^4\tau K_2(K_1+\sigma f_{min})}$.

Proof. Proceeding exactly as in Theorem 3, we can easily prove the theorem.

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On Exchange Principle in Triply Diffusive Convection Analogous to Stern Type in Porous Medium

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Abstract

Condition for characterizing non oscillatory motions, which may be neutral or unstable, for triply diffusive convection analogous to Stern type in a porous medium is derived by using Darcy-Brinkman model. It is analytically proved that the principle of the exchange of stabilities, in triply diffusive convection in a porous medium, is valid in the regime $\frac{|R|E\sigma}{2\pi^4} \leq 1$, where R is the thermal Rayleigh number, σ is the Prandtl number, E is a constant. It is further proved that this result is uniformly valid for all combinations of rigid and dynamically free boundaries.

Keywords: Triply diffusive convection, Porous medium, Darcy-Brinkman model, The principle of the exchange of stabilities, Concentration Rayleigh number.

Introduction

Research on convective fluid motion in porous media under the simultaneous action of a uniform vertical temperature gradient and a gravitationally opposite uniform vertical concentration gradient (known as double diffusive convection) has been an area of great activity due to its importance in the predication of ground water movement in aquifers, in assessing the effectiveness of fibrous materials, in engineering geology and in nuclear engineering. Double diffusive convection is now well known. For a broad view of the subject one may be referred to Nield and Bezan [12], Murray and Chen [10], Nield [11], Taunton et al. [29], Kuznetsov and Nield [8], Lombardo and Mulone [9], Basu and Layek[2].

All these researchers have considered double diffusive convection. However, it has been recognized later that there are many fluid systems, in which more than two components are present. For example, Degens et al. [3] reported that the saline waters of geothermally heated Lake kivu are strongly stratified by heat and a salinity which is the sum of comparable concentrations of many salts. Similarly the oceans contain many salts having concentrations less than a few percent of the sodium chloride concentration. Multi-component concentrations can also be found in magmas and substratum of water reservoirs. The subject with more than two components (in porous and non porous medium) has attached the attention of many researchers Grifiths [4, 5], Poulikakos [15], Pearlstein et al. [14], Terrones and Pearlstein [26], Rudraiah and Vortmeyer [20], Lopez et al. [7], Tracey [27, 28], Rionero [17], Straughan and Tracey [24]. The essence of the works of these researchers is that small salinity of a third component with a smaller mass diffusivity can have a significant effect upon the nature of convection; and ‘oscillatory’ and direct ‘salt finger’ modes are simultaneous possible under a wide range of conditions, when the density gradients due to components with greatest and smallest diffusivity are of same signs. Terrones [25] studied the effects of cross-diffusion on the onset of convective instability in a horizontally infinite triply diffusive and triply stratified fluid layer. Ryzhkov and Shevtsova [21] investigated the long-wave instability of a vertical multicomponent fluid layer induced by the Soret effect. Rionero [18] investigated a triply convective diffusive fluid mixture saturating a porous layer and derived sufficient conditions for inhibiting the onset of convection. Rionero [19] further studied the multicomponent diffusive convection in porous layer salted by m salts partly from above and partly from below.

The validity of the principle of the exchange of stabilities (PES) (i.e. nonoccurence of oscillatory motions) in stability problems removes the unsteady terms from the linear perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is defined by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and southwell [13] proved the validity of PES for Rayleigh-Benard problem. However no such result exists for other more complex hydrodynamic configurations. Banerjee et al. [1] derived a sufficient condition for the validity of PES for hydromagnetic Rayleigh-Benard problem. Gupta et al. [6] extended Banerjee et al.’s [1] criterion to rotatory hydromagnetic thermohaline convection problem. To the author’s knowledge no such result exists for triply diffusive convection analogous to Stern [23] type in porous medium. Thus the present paper provides a sufficient condition for the validity of PES in triply diffusive convection analogous to Stern [23] type in porous medium may be regarded as a first step in this scheme of extended investigations. The following result is obtained in this direction:

For triply diffusive convection in porous medium, if $\frac{|R|E\sigma}{2\pi^4} \leq 1$, then an arbitrary neutral or unstable mode of system is definitely nonoscillatory in character and in particular PES is valid where R is the Raleigh number, σ is the Prandtl number, E is a constant. It is further proved that this result is uniformly valid for all combinations of rigid and dynamically free boundaries and the results for Rayleigh-Benard convection in porous medium and double diffusive convection of Stern [23] type in porous medium follow as a consequence

Mathematical Formulation and Analysis

A viscous finitely heat conducting Boussinesq fluid layer, saturating a porous medium, of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and $T_1 (> T_0)$ and uniform concentrations S_{10}, S_{20} and $S_{11} (> S_{10}), S_{21} (> S_{20})$ (as shown in Fig.1). It is assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Darcy- Brinkman model has been used to investigate the triple diffusive convection in porous medium.

Non-dimensional hydrodynamical equations that govern the problem are given by Vafai [30], Prakash et al. [16]

$$\Lambda(D^2 - a^2)^2 w - (p + D_a^{-1})(D^2 - a^2)w = -|R| a^2 \int_0^1 w^* \theta \, dz + |R_1| a^2 \int_0^1 w^* \square_1 \, dz + |R_2| a^2 \int_0^1 w^* \square_2 \, dz. \quad (1)$$

$$(D^2 - a^2 - E\,\sigma p)\theta = -w, \quad (2)$$

$$\left(D^2 - a^2 - \frac{E_1 \sigma p}{\tau_1}\right) \square_1 = -\frac{w}{\tau_1}, \quad (3)$$

$$\left(D^2 - a^2 - \frac{E_2 \sigma p}{\tau_2}\right) \square_2 = -\frac{w}{\tau_2}. \quad (4)$$

The equations (1) – (4) are to be solved by using the following boundary conditions:

$w = \theta = \square_1 = \square_2 = Dw = 0$ at $z = 0$ and at $z = 1$, (when both the boundaries are rigid) (5)

or $w = \theta = \square_1 = \square_2 = D^2 w = 0$ at $z = 0$ and at $z = 1$, (when both the boundaries are free) (6)

$$\left.\begin{array}{ll} \text{or} & w = \theta = \square_1 = \square_2 = Dw = 0 \text{ at } z = 0, \text{ (when lower boundary is rigid)} \\ \text{and} & w = \theta = \square_1 = \square_2 = D^2 w = 0 \text{ at } z = 1, \text{ (when upper boundary is free)} \end{array}\right\} \quad (7)$$

$$\left.\begin{array}{ll} \text{or} & w = \theta = \square_1 = \square_2 = D^2 w = 0 \text{ at } z = 0, \text{ (when lower boundary is free)} \\ \text{and} & w = \theta = \square_1 = \square_2 = Dw = 0 \text{ at } z = 1, \text{ (when upper boundary is rigid)} \end{array}\right\} \quad (8)$$

where z is the real independent such that $0 \le z \le 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $\sigma = \frac{\nu\square}{\kappa}$ is the Prandtl number, $\tau_1 = \frac{\kappa_1}{\kappa}$ and $\tau_2 = \frac{\kappa_2}{\kappa}$ are the Lewis numbers, $R = \frac{g\,\alpha\,\beta\,d^4}{\kappa\,\nu}$ is the thermal Rayleigh number, $R_1 = \frac{g\,\alpha_1 \beta_1\,d^4}{\kappa\,\nu}$ and $R_2 =$

$\frac{g\,\alpha_2\beta_2\,d^4}{\kappa\,\nu}$ are the two concentration Rayleigh numbers, $p = p_r + ip_i$ is the complex growth rate where p_r and p_i are the real constants, w is the vertical velocity, θ, is the temperature, ϕ_1 and ϕ_2 are the two concentrations. It may further be noted that in Eqs. (1)-(4) together with the boundary conditions (5)-(8) describe an eigenvalue problem for p and govern triply diffusive convection in porous medium for any combination of dynamically free and rigid boundaries.

Now we prove the following theorem

Theorem. If $(w, \theta, \phi_1, \phi_2, p)$, $p = p_r + ip_i$, $p_r \geq 0$ is a solution of Eqs. (1) – (8) with $R < 0$, $R_1 < 0$, $R_2 < 0$ and $\frac{|R|E\sigma}{2\pi^4} \leq 1$ then $p_i = 0$. In particular $p_r = 0$ implies $p_i = 0$, if $\frac{|R|E\sigma}{2\pi^4} \leq 1$.

Proof: Multiplying equation (1) by w* (the superscript * henceforth denotes complex conjugation) on both sides and integrating over vertical range of z, we obtain

$$\Lambda\int_0^1 w^*\,(D^2-a^2)^2 w\,dz - (p + D_a^{-1})\int_0^1 w^*\,(D^2-a^2)w\,dz = -|R|a^2\int_0^1 w^*\,\theta\,dz + |R_1|a^2\int_0^1 w^*\,\phi_1\,dz + |R_2|a^2\int_0^1 w^*\,\phi_2\,dz. \tag{9}$$

Making use of Eqs. (2) – (4) and the fact that w (0) = 0 = w (1), we can write

$$|R|\,a^2\int_0^1 w^*\,\theta\,dz = |R|a^2\int_0^1 \theta\,(D^2-a^2-E\,\sigma\,p^*)\theta^* dz, \tag{10}$$

$$|R_1|a^2\int_0^1 w^*\,\phi_1 dz = -|R_1|a^2\tau_1\int_0^1 \phi_1\left(D^2-a^2-\frac{E_1\sigma\,p^*}{\tau_1}\right)\phi_1^* dz, \tag{11}$$

$$|R_2|a^2\int_0^1 w^*\,\phi_2 dz = -|R_2|a^2\tau_2\int_0^1 \phi_2\left(D^2-a^2-\frac{E_2\sigma\,p^*}{\tau_2}\right)\phi_2^* dz. \tag{12}$$

Combining Eqs. (9) – (12), we obtain

$$\Lambda\int_0^1 w^*\,(D^2-a^2)^2 w\,dz - (p + D_a^{-1})\int_0^1 w^*\,(D^2-a^2)w\,dz = |R|a^2\int_0^1 \theta\,(D^2-a^2-E\,\sigma\,p^*)\theta^* dz - |R_1|a^2\tau_1\int_0^1 \phi_1\left(D^2-a^2-\frac{E_1\sigma\,p^*}{\tau_1}\right)\phi_1^* dz - |R_2|a^2\tau_2\int_0^1 \phi_2\left(D^2-a^2-\frac{E_2\sigma\,p^*}{\tau_2}\right)\phi_2^* dz. \tag{13}$$

Integrating various terms of equation (13), by parts, for an appropriate number of times and making use of either of the boundary conditions (5) – (8), it follows that

$$\Lambda \int_0^1 (|D^2w|^2 + 2a^2|Dw|^2 + a^4|w|^2)\ dz + (p + D_a^{-1}) \int_0^1 (|Dw|^2 + a^2|w|^2)\, dz =$$
$$-|R|a^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + E\sigma p^*|\theta|^2)\, dz + |R_1|a^2\tau_1 \int_0^1 \Big(|D\square_1|^2 + a^2|\square_1|^2 +$$
$$\frac{E_1\sigma\, p^*}{\tau_1}|\square_1|^2\Big)\, dz + |\, R_2|a^2\tau_2 \int_0^1 \Big(|D\phi_2|^2 + a^2|\phi_2|^2 + \frac{E_2\sigma\, p^*}{\tau_2}|\phi_2|^2\Big)\, dz. \tag{14}$$

Equating imaginary parts on both sides of equation (14) and cancelling $p_i(\neq 0)$ throughout, we have

$$\int_0^1 (|Dw|^2 + a^2|w|^2)\, dz = |R|a^2E\sigma \int_0^1 |\theta|^2 dz - |R_1|a^2E_1\sigma \int_0^1 |\square_1|^2 dz -$$
$$|R_2|a^2E_2\sigma \int_0^1 |\square_2|^2 dz\,. \tag{15}$$

Now, multiplying equation (2) by its complex conjugate and integrating the resulting equation for a suitable number of times and use the boundary condition on θ namely, $\theta(0) = 0 = \theta(1)$, we obtain

$$\int_0^1 (|D^2\theta|^2 + 2a^2|D\theta|^2 + a^4|\theta|^2)\ dz + 2E\sigma p_r \int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz +$$
$$E^2\sigma^2|p|^2 \int_0^1 |\theta|^2 dz = \int_0^1 |w|^2 dz. \tag{16}$$

Since $p_r \geq 0$, it follows from equation (16), that

$$2a^2 \int_0^1 |D\theta|^2 dz < \int_0^1 |w|^2 dz. \tag{17}$$

Now, since θ and w satisfy the boundary conditions $\theta(0) = 0 = \theta(1)$ and $w(0) = 0 = w(1)$ respectively, we have by Rayleigh-Ritz inequality (Schultz [22])

$$\int_0^1 |D\theta|^2 dz \geq \pi^2 \int_0^1 |\theta|^2 dz, \tag{18}$$

and $$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \tag{19}$$

Utilizing inequality (18) and (19) in inequality (17), we get

$$a^2 \int_0^1 |\theta|^2 dz < \frac{1}{2\pi^4} \int_0^1 |Dw|^2 dz. \tag{20}$$

Utilizing inequality (20) in Eq. (16), we obtain

$$\left[1 - \frac{|R|E\sigma}{2\pi^4}\right] \int_0^1 |Dw|^2 dz +$$
$$a^2 \int_0^1 |w|^2 dz + |R_1|a^2E_1\sigma \int_0^1 |\square_1|^2 dz + |R_2|a^2E_2\sigma \int_0^1 |\square_2|^2 dz < 0\,. \tag{21}$$

which, clearly implies that

$$\frac{|R|E\sigma}{2\pi^4} > 1. \tag{22}$$

Hence if $\frac{|R|E\sigma}{2\pi^4} \leq 1$, then we must have $p_i = 0$.

This proves the theorem.

The essential content of the theorem from the physical point of view are that for the problem of triply diffusive convection analogous to Stern type in porous medium, an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular the principle of the exchange of stabilities is valid if $\frac{|R|E\sigma}{2\pi^4} \leq 1$. Further this result is uniformly valid for any combination of rigid and / or free boundaries.

Special Cases: It follows from theorem1 that an arbitrary neutral or unstable mode is non oscillatory in character and in particular PES is valid for:

1. Rayleigh-Benard convection in porous medium ($R_1 = R_2 = 0$).
2. Thermohaline convection of Stern (1960) type in porous medium ($R < 0, R_1 < 0, R_2 < 0$) if $\frac{|R|E\sigma}{2\pi^4} \leq 1$.

Conclusion

Linear stability theory is used to derive a sufficient condition for the validity of the 'PES' in triply diffusive convection in porous medium. It is further proved that this result is uniformly valid for any combination of rigid and / or free boundaries.

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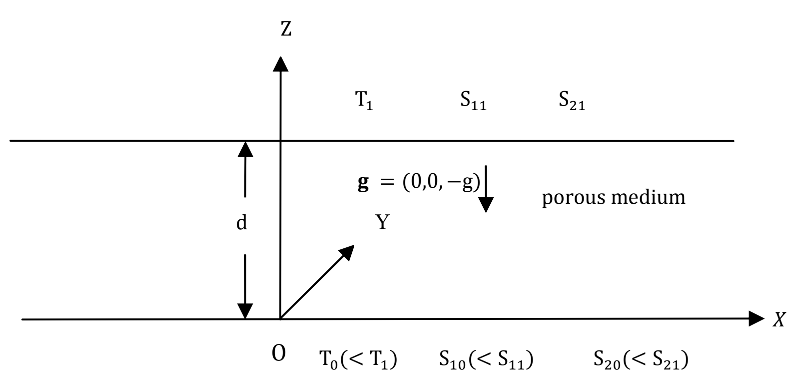

Fig. 1. Physical Configuration.

Thermal instability in a porous medium layer saturated by a viscoelastic fluid in electrohydrodynamics: Brinkman model

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Abstract

Thermal instability in a horizontal layer of a porous medium saturated by viscoelastic fluid in electrohydrodynamics is studied both analytically and numerically. Rivlin-Ericksen fluid model is used to describe the behaviour of a viscoelastic fluid and for the porous medium, Brinkman model is employed. The fluid layer is induced by the dielectrophoretic force due to the variation of dielectric constant with temperature. By applying perturbation solutions and linear stability theory, we derive the dispersion relation describing the influence of viscolasticity, Brinkman-Darcy number, Darcy number and electric Rayleigh number. It is observed that Rivlin-Ericksen viscoelastic fluid behaves like an ordinary Newtonian fluid in the stationary convection. The effects Brinkman-Darcy number, Darcy number and AC electric field studied both analytically and numerically for free-free boundaries on the stationary convection. The present results are in good agreement with the earlier published results.

Key words: Rivlin-Ericksen fluid, AC electric field, Viscosity, Viscoelasticity, Porous medium.

1. Introduction

Electrohydrodynamics (EHD) can be regarded as branch of fluid mechanics which deal with the dynamics of electrically charged fluids, also known as electro-fluid-dynamics (EFD) or electrokinetics. EHD covers the fluid transport mechanisms such as electrophoresis, electrokinetics, dielectrophoresis, electro-osmosis, and electrorotation.

Recently, the study of electrohydrodynamic instability in dielectric fluid attracts many researchers because it has various applications in climatology, oceanography, EHD enhanced thermal transfer, EHD pumps, EHD in microgravity, micromechanic systems, drug delivery, micro-cooling system, nanotechnology etc. Chen et al. [1] discussed the applications of electrohydrodynamics in brief. They said that EHD heat transfer came out as an alternative method to enhance heat transfer, which is known as electrothermohydrodynamics (ETHD). Many researchers have been studied the effect of AC or DC electric field on natural convection in a horizontal dielectric fluid layer by taking different types of fluids. The onset of electrohydodynamic convection in a horizontal layer of dielectric fluid was studied by Landau [2], Robert [3], Castellanos [4], Lin [5], Gross and Porter [6], Turnbull [7], Maekawa et al. [8], Smorodin and Velarde [9], Galal [10], Rudraiah and Gayathri [11] and Chang et al. [12]. Takashima and Ghosh [13] studied the electrohydrodynamic instability in a viscoelastic liquid layer and found that oscillatory modes of instability exist only when the thickness of the liquid layer is smaller than about 0.5 mm and for such a thin layer the force of electrical origin is much more important than buoyancy force while Takashima and Hamabata [14] studied the stability of natural convection in a vertical layer of dielectric fluid in the presence of a horizontal AC electric field.

The study of Newtonian fluid heated from below saturating a porous medium has attracted many researchers for the last few decades since it has various applications in geophysics, food processing, oceanography, soil sciences, ground water hydrology and astrophysics etc. Chandrasekher [15] discussed in detail the thermal instability of Newtonian fluid under the various assumptions of hydrodynamics and hydromagnetics. A good account of thermal instability problems in a porous medium is given by Wooding [16], Ingham and Pop [17], Vafai and Hadim [18] and Nield and Bejan [19].

Reiner [20] and Rivlin and Ericksen [21] developed the non-linear constitutive equations for non-Newtonian compressible and incompressible fluid respectively. Green [22] was the first who studied the problem of convective instability of a viscoelastic fluid heated from below while Vest and Arpaci [23] studied the problem of overstability of a viscoelastic fluid. With the growing importance of non-Newtonian fluids having applications in geophysical fluid dynamics, chemical technology and petroleum industry attracted widespread interest in the study on non-Newtonian fluids. There are many common materials such as paints, polymers, coolants, plastics, magma, saturated soils and Earth's lithosphere which behave as viscoelastic fluid. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or by Oldroyd's constitutive relations. One such type of fluids is Rivlin-Ericksen viscoelastic fluid having relevance in chemical technology and industry. Rivlin-Ericksen viscoelastic fluid forms the basis for the manufacture of many important polymers and useful products. Such polymers are used in agriculture, communication appliances and in bio medical applications. Examples of these applications are filtration processes, packed bed reactors, insulation system, ceramic processing, enhanced oil recovery, chromatography.

In the case of Rivlin-Ericksen fluid, the term $\left[\mu\nabla^2\mathbf{q}\right]$ in the equations of motion is replaced by the term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\mathbf{q}\right]$, where μ and μ' are the viscosity and viscoelasticity of the incompressible Rivlin-Ericksen fluid, k_1 is the medium permeability and **q** is the Darcian (filter) velocity of the fluid. Also the constitutive equation is one of the simplest viscoelastic laws that accounts for normal stress effects responsible for the periodic phenomena arising in viscoelastic fluids. Because of these reasons, the model has been widely accepted for experimental measurements and flow visualization on the instability of viscoelastic flows. A good account of thermal instability problems of Rivlin-Ericksen fluid in porous medium has been studied by Sharma et al. [25], Rana and Thakur [26], Chand and Rana [27], Rana and Sharma [28] and Chand et al. (2015).

Shivakumara et al. [29] studied the electrothermoconvection in a rotating Brinkman porous layer while Rana et al. [30] studied the electrohydrodynamic instability of Rivlin-Ericksen viscoelastic dielectric fluid layer. In the present paper thermal instability in a Brinkman porous medium layer saturated by a viscoelastic fluid in electrohydrodynamics is studied which include an additional parameter Brinkman-Darcy number. The Darcy-Brinkman equation is a governing equation for flow through a porous medium with an extra Laplacian (viscous) term (Brinkman term) is added to the classical Darcy equation. The equation has been widely applied to examine high-porosity porous media.

2. **Theoretical Model and Mathematical Analysis**

We consider an infinite horizontal layer of an incompressible Rivlin-Ericksen viscoelastic fluid of thickness d saturating a porous medium, bounded by the planes z = 0 and z = d as shown in **fig.1**. The fluid layer is acted upon by a gravity force **g =** (0, 0, -g) aligned in the z direction and the uniform vertical AC electric field applied across the layer. The temperature T at the lower and upper boundaries is assumed to take constant values T_0 and T_1 ($< T_0$) respectively. The Darcy-Brinkman law is assumed to hold and the Oberbeck-Boussinesq approximation is employed.

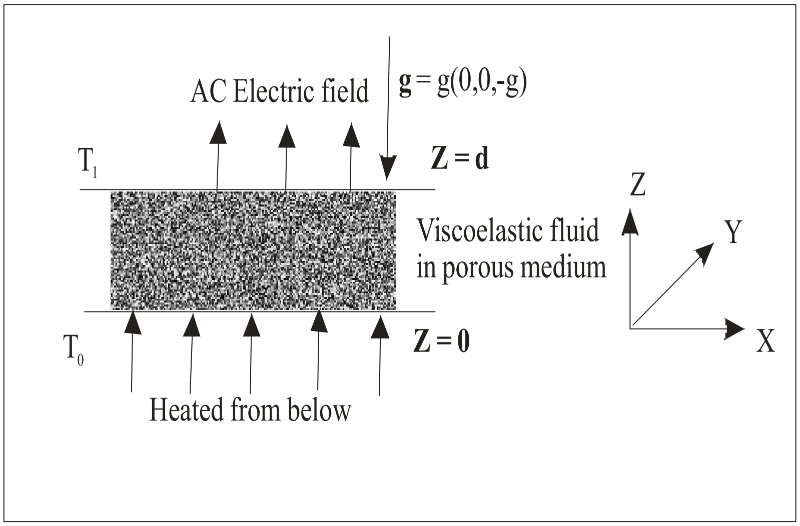

Fig. 1 Physical configuration

2.1 Governing Equations

Let $\rho, \mu, \tilde{\mu}, \mu^{'}, \phi, p, K$, $\mathbf{q}$(u, v, w), $\mathbf{g}$, T, κ, A and $\mathbf{E}$ denote respectively, the density, viscosity, effective viscosity, viscoelasticity, medium porosity, pressure, dielectric constant, Darcy velocity vector, acceleration due to gravity, temperature, thermal diffusivity, ratio of heat capacity and the root-mean-square value of electric field. The equations of conservation of mass, momentum and thermal energy for Rivlin-Ericksen elastico-viscous fluid (Chandrasekhar [15], Rivlin-Ericksen [21], Takashima and Ghosh [13], Rana and Sharma [27], Shivakumara [29] and Rana et al. [30]) are

$$\nabla \cdot \mathbf{q} = 0, \tag{1}$$

$$\frac{\rho}{\phi}\frac{d\mathbf{q}}{dt} = -\nabla P + \rho\mathbf{g} + \tilde{\mu}\nabla^2\mathbf{q} - \frac{1}{k_1}\left(\mu + \mu^{'}\frac{\partial}{\partial t}\right)\mathbf{q} - \frac{1}{2}(\mathbf{E}\cdot\mathbf{E})\nabla K, \tag{2}$$

$$A\frac{\partial T}{\partial t} + (\mathbf{q}\cdot\nabla)T = \kappa\nabla^2 \mathrm{T}, \tag{3}$$

where $\frac{\mathrm{d}}{\mathrm{dt}} = \frac{\partial}{\partial \mathrm{t}} + \frac{1}{\varepsilon}(\mathbf{q}\cdot\nabla)$ stands for convection derivative

and $$\mathrm{P} = p - \frac{\rho}{2}\frac{\partial K}{\partial \rho}(\mathbf{E}\cdot\mathbf{E}) \tag{4}$$

is the modified pressure.

The Coulomb force term $\rho_e \mathbf{E}$, where ρ_e is the free charge density, is of negligible order as compared with the dielectrophoretic force term for most dielectric fluids in a 60Hz AC electric field. Thus, we retain only the dielectrophoretic term, i. e. last term in equation (2) and neglect the Coulomb force term. Furthermore, the electrical relaxation times of most dielectric liquids appear to be sufficient long to prevent the build up of free charge at standard power line frequencies. At the same time, dielectric loss at these frequencies is very low that it makes no significant contribution to the temperature field. It is also seen that the dielectrophoretic force term depends on $(\mathbf{E} \cdot \mathbf{E})$rather than **E**. As the variation of **E** is so speedy, the root-mean-square value of **E** is used as effective value in determining the motion of fluids. So we can consider the AC electric field as the Dc electric field whose strength is equal to the root mean square value of the AC electric field.

A charged body in an electric field tends to along the electric field lines and impart momentum to the surrounding fluid. The Maxwell equations are

$$\nabla \times \mathbf{E} = 0\,, \tag{5}$$

$$\nabla \cdot (K\mathbf{E}) = 0\,. \tag{6}$$

In view of Eq. (5), **E** can be expressed as

$$\mathbf{E} = -\nabla V\,, \tag{7}$$

where V is the root mean square value of electric potential. The dielectric constant is assumed to be linear function of temperature and is of the form

$$K = K_0[1 - \gamma(T - T_0)], \tag{8}$$

where $\gamma > 0$, is the thermal coefficient of expansion of dielectric constant and is assumed to be small.

The equation of state is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{9}$$

where α is coefficient of thermal expansion and the suffix zero refers to values at the reference level z = 0.

2.2 Basic State

The basic state of the system is taken to be quiescent layer (no settling) and is given by

$$\mathbf{q} = \mathbf{q}_b(z), P = P_b(z), T = T_b(z), \mathbf{E} = \mathbf{E}_b(z), K = K_b(z), \rho = \rho_b(z)\,, \tag{10}$$

where the subscript b denotes the basic state.

Substituting equations given in (10) in Eqs. (1) – (9), we obtain

$$0 = -\nabla \frac{P_b(z)}{\rho_0} + \frac{\rho_b(z)}{\rho_0}\mathbf{g} - \frac{1}{2\rho_0}\left(\mathbf{E}^2\right)\nabla K, \tag{11}$$

$$\frac{d^2 T_b(z)}{dz^2} = 0, \tag{12}$$

$$K_b(z) = K_0\left[1 - \gamma\left(T_b - T_0\right),\right] \tag{13}$$

$$\rho_b(z) = \rho_0\left[1 - \alpha\left(T_b - T_0\right)\right], \tag{14}$$

$$\nabla.\left(K_b E_b\right) = 0. \tag{15}$$

Solving Eq. (12) by using the following boundary conditions

$$T_b(z) = T_0 \text{ at z = 0 and } T_b(z) = T_1 \text{ at z = 1} \tag{16}$$

we obtain

$$T_b = T_0 - \Delta T z / d. \tag{17}$$

In view of Eq. (15) and noting that $E_{bx} = E_{by} = 0.$ It follows that

$$K_b E_{bz} = K_0 E_0 = \text{constant (say).} \tag{18}$$

Then

$$\mathbf{E} = \mathbf{E}_b(z) = \frac{E_0}{1 + \gamma \Delta T z / d}. \tag{19}$$

Hence $V_b(z) = -\dfrac{E_0 d}{\gamma \Delta T}\log\left(1 + \gamma \Delta T z / d\right),$ (20)

where $E_0 = -\dfrac{V_1 \gamma \Delta T / d}{\log\left(1 + \gamma \Delta T\right)}$ (21)

is the root-mean-square value of the electric field at z = 0.

2.3 Perturbation Solutions

To study the stability of the system, we superimposed infinitesimal perturbations on the basic state, so that

$$\mathbf{q} = \mathbf{q}', T = T_b + \mathrm{T}', \ \mathbf{E} = \mathbf{E}_b + \mathbf{E}', \rho = \rho_b + \rho', \mathrm{K} = \mathrm{K_b} + K', P = P_b + P' \tag{22}$$

where

$\mathbf{q}', \mathrm{T}', \mathbf{E}', \rho', K', P'$ be the perturbations in $\mathbf{q}, T, \mathbf{E}', \rho, K', P'$ respectively. Substituting Eq. (10) in Eqs. (1) – (9), linearizing the equations by neglecting the product of primed quantities, eliminating the pressure from the momentum Eq. (2) by operating curl twice and retaining the vertical component and non-dimensionalising the resulting equations by introducing the dimensionless variables as follows:

$$(\mathrm{x}', \mathrm{y}', \mathrm{z}',) = \left(\frac{\mathrm{x, y, z}}{\mathrm{d}}\right), \ \mathbf{q}' = \frac{d}{\kappa}\mathbf{q}, \ \mathrm{t}' = \frac{\kappa}{\mathrm{d}^2}\mathrm{t}, \ \mathrm{T}' = \frac{1}{\Delta T}\mathrm{T}, \ \mathrm{V}' = \frac{1}{\gamma \mathrm{E}_0 \Delta T d}V$$

Neglecting the primes for simplicity, we obtain the linear stability equations in the form

$$\left[\frac{1}{\Pr}\frac{\partial}{\partial t} + \frac{1}{Da}\left(1 + F\frac{\partial}{\partial t}\right) - \widetilde{D}a\nabla^2\right]\nabla^2 w = Ra_t \nabla_h^2 T + Ra_e \nabla_h^2\left(T - \frac{\partial V}{\partial z}\right), \tag{23}$$

$$\left[\frac{\partial}{\partial t} - \nabla^2\right]T = w, \tag{24}$$

$$\nabla^2 V = \frac{\partial T}{\partial z}, \tag{25}$$

where we have used dimensionless parameters as:

$$\Pr = \frac{\nu\phi}{\kappa},$$

$$F = \frac{\mu'}{\mu},$$

$$Da = \frac{k_1}{d^2},$$

$$\widetilde{D}a = \frac{\widetilde{\mu}k_1}{\mu d^2}, \qquad \text{26a, b, c, d)}$$

$$Ra_t = \frac{g\alpha\Delta T d^3}{\nu\kappa}, \tag{27}$$

$$Ra_e = \frac{\gamma^2 K_0 E_0^2 (\Delta T)^2 d^2}{\mu\kappa}, \tag{28}$$

The parameter Pr is the Prandtl number, F is the viscoelasticity parameter, Da is the Darcy number, Ra_t is the familiar thermal Rayleigh number and Ra_e is the AC electric Rayleigh number.

Now we assume that the temperature at the boundaries is kept fixed, the fluid layer is confined between two boundaries. The boundary conditions appropriate (Chandrasekhar [15], Takashima and Ghosh [13], Rana and Sharma [27] and Rana et al. [30]) to the problem are

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial V}{\partial z} = 0, T = 0 \text{ or DT} = 0.$$

(29)

2. Linear stability analysis

Using normal mode analysis method, we assume that the perturbation quantities have x, y and t dependence of the form

$$[w, T, \mathrm{V}] = [W(z), \Theta(z), \Phi(z)]\exp(ilx + imy + \omega t), \tag{30}$$

where l and m are the wave numbers in the x and y direction, respectively, and ω is the complex growth rate of the disturbances.

Substituting Eq. (30) in Eqs. (23) – (25) and (29), we get

$$\left[\frac{\omega}{\Pr} + \frac{1}{Da}(1 + F\omega) - \widetilde{D}a\left(D^2 - a^2\right)\right]\left(D^2 - a^2\right)W = -Ra_t a^2\Theta + Ra_e a^2(\Theta - D\Phi), \tag{31}$$

$$\left[A\omega - \left(D^2 - a^2\right)\right]\Theta = W, \tag{32}$$

$$\left(D^2 - a^2\right)\Phi = D\Theta, \tag{33}$$

$$W = D^2W = D\Phi = 0, \Theta = 0 \text{ or } D\Theta = 0, \tag{34}$$

where $a^2 = l^2 + m^2, D = \frac{d}{dz}$.

Eqs. (31) – (33) form an eigenvalue problem for Ra_t or Ra_e and ω with respect to the boundary conditions (34).

We assume the solution to W, Θ, Φ and Z of the form

$$W = W_0 \sin \pi z, \ \Theta = \Theta_0 \sin \pi z, \ \Phi = \Phi_0 \cos \pi z, \tag{35}$$

which satisfy the boundary conditions of Eq. (34). Substituting Eq. (35) into Eqs. (31) – (33), we obtain the following matrix equation

$$\begin{bmatrix} \left[\frac{\omega}{\Pr}+\frac{1}{Da}(1+\omega F)+\tilde{D}aJ^2\right]J^2 & -a^2(Ra_t+Ra_e) & -Ra_e a^2\pi \\ -1 & A\omega+J^2 & 0 \\ 0 & \pi & J^2 \end{bmatrix}\begin{bmatrix} W_0 \\ \Theta_0 \\ \Phi_0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{36}$$

where $J^2=\pi^2+a^2$ is the total wave number.

The linear system (36) has a non-trivial solution if and only if

$$\begin{vmatrix} \left[\frac{\omega}{\Pr}+\frac{1}{Da}(1+\omega F)+\tilde{D}aJ^2\right]J^2 & -a^2(Ra_t+Ra_e) & -Ra_e a^2\pi \\ -1 & A\omega+J^2 & 0 \\ 0 & \pi & J^2 \end{vmatrix}=0,$$

which yields

$$Ra_t=\frac{J^2(J^2+A\omega)}{a^2}\left[\frac{\omega}{\Pr}+\frac{1}{Da}(1+\omega F)+\tilde{D}aJ^2\right]-\frac{a^2}{J^2}Ra_e. \tag{37}$$

Eq. (37) is the dispersion relation accounting for the effect of Prandtl number, electric Rayleigh number, Darcy number, Brinkman-Darcy number and kinematic viscoelasticity parameter in a layer of Rivlin-Ericksen viscoelastic dielectric fluid in porous medium.

3. Stationary convection

For stationary convection, putting $\omega = 0$ in equation (37) reduces it to

$$Ra_t=\frac{(\pi^2+a^2)^3\tilde{D}a+(\pi^2+a^2)^2Da^{-1}}{a^2}-\frac{a^2}{\pi^2+a^2}Ra_e. \tag{38}$$

Eq. (38) expresses the thermal Rayleigh number as a function of the dimensionless resultant wave number a, the parameters electric Rayleigh number Ra_e and Darcy number Da. It is found that the kinematic viscoelasticity parameter F vanishes with ω and the

Rivlin-Ericksen viscoelastic dielectric fluid behaves like an ordinary Newtonian dielectric fluid. Eq. (38) is in good agreement with the equation obtained by Roberts [3].

In the absence of AC electric field (i. e., when $Ra_e = 0$), Eq. (38) reduces to

$$Ra_t = \frac{\left(\pi^2 + a^2\right)^3 \widetilde{D}a + \left(\pi^2 + a^2\right)^2 Da^{-1}}{a^2}. \tag{39}$$

To study the effect of AC electric field on electrohydrodynamic stationary convection, we examine the behaviour of $\frac{\partial Ra_t}{\partial Ra_e}$, $\frac{\partial Ra_t}{\partial Da}$, $\frac{\partial Ra_t}{\partial \widetilde{D}a}$ analytically and numerically.

From Eq. (38), we obtain

$$\frac{\partial Ra_t}{\partial Ra_e} = -\frac{a^2}{\pi^2 + a^2}, \tag{40}$$

which is negative implying thereby AC electric field has destabilizing effect on the system which is in an agreement with the results derived by Takashima and Ghosh [24], Shivakumara et al. [29] and Rana et al. [30].

Also Eq. (38) yields

$$\frac{\partial Ra_t}{\partial Da} = -\frac{\left(\pi^2 + a^2\right)^2 Da^{-2}}{a^2}, \tag{41}$$

which is negative implying thereby Darcy number has destabilizing effect on the system which is in good agreement with the results derived by Rana and Sharma [27], Rana and Thakur [25], Shivakumara et al. [29] and Rana et al. [30].

From Eq. (38), we get

$$\frac{\partial Ra_t}{\partial \widetilde{D}a} = \frac{\left(\pi^2 + a^2\right)^3}{a^2}, \tag{42}$$

which is positive implying thereby Brinkman-Darcy number has stabilizing effect on the system which is in good agreement with the results derived by Chand and Rana [26], Shivakumara et al. [29].

The dispersion relation (38) is analysed numerically. Graphs have been plotted by giving some numerical values to the parameters, to depict the stability characteristics.

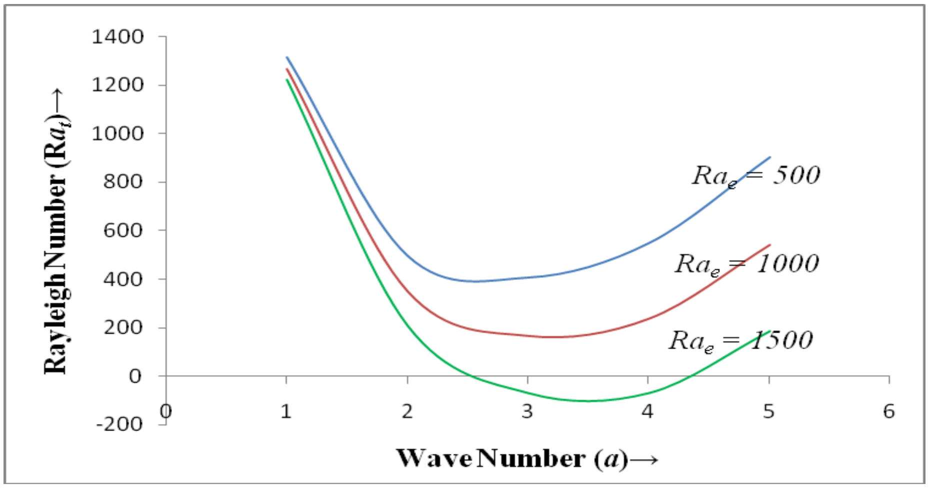

Fig. 2: Variation of thermal Rayleigh number (Ra_t) with wave number (a) for different values of electric Rayleigh number (Ra_e).

In fig.2, the thermal Rayleigh number Ra_t is plotted against dimensionless wave number a for different values of electric Rayleigh number Ra_e as shown. This shows that as (Ra_e) increases the thermal Rayleigh number Ra_t decreases. Thus, AC electric field has destabilizing effect on stationary convection which is in good agreement with the result obtained analytically in Eq. (40).

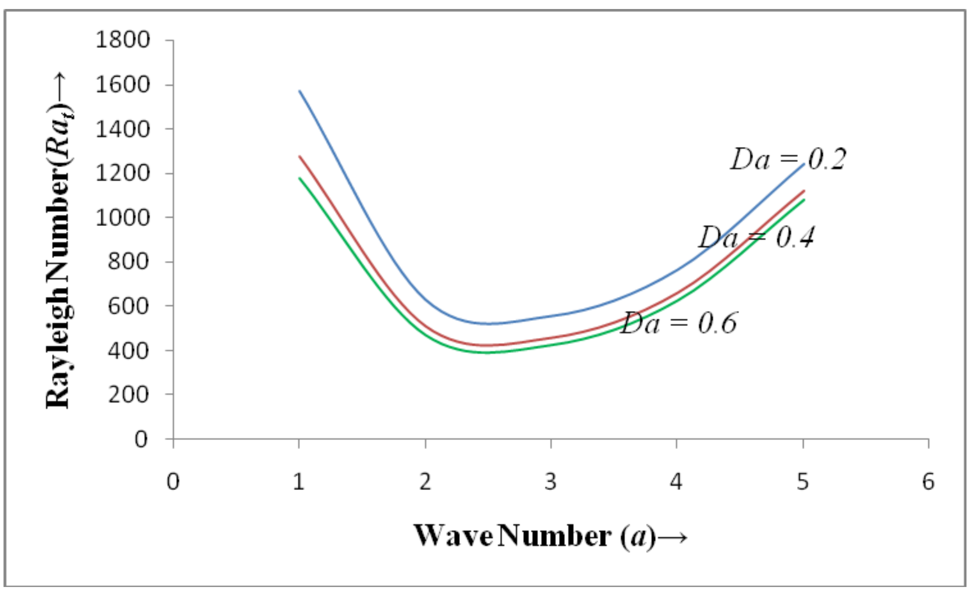

Fig. 3: Variation of thermal Rayleigh number (Ra_t) with wave number (a) for different values of Darcy number (Da)

In fig.3, the thermal Rayleigh number Ra_t is plotted against dimensionless wave number a for different values of Darcy number Da as shown. This figure depicts that as Darcy number Da increases the thermal Rayleigh number Ra_t decreases. Therefore, Darcy number has destabilizing effect on the stationary convection which is in good agreement with the result obtained analytically in Eq. (41).

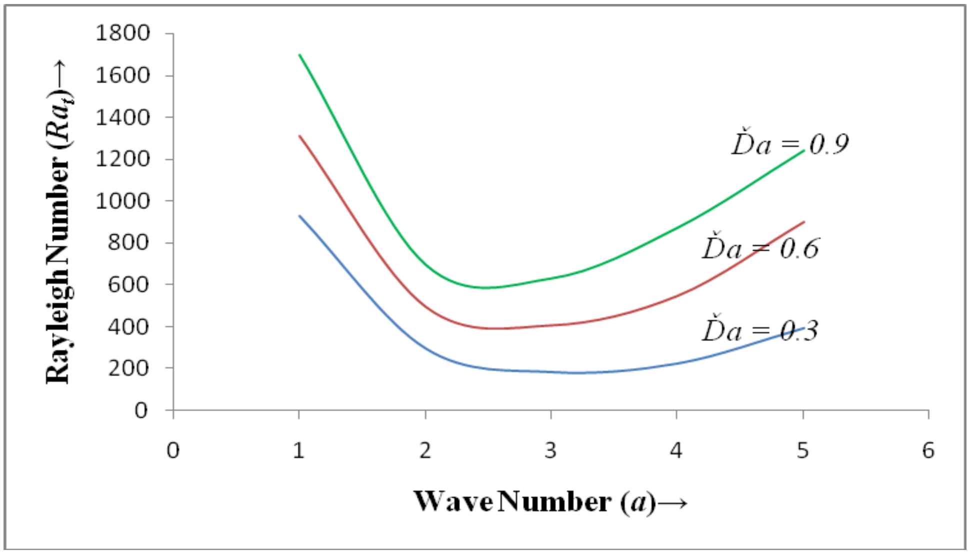

Fig. 4: Variation of thermal Rayleigh number (Ra_t) with wave number (a) for different values of Brinkman-Darcy Number ($\tilde{D}a$)

In fig.4, the thermal Rayleigh number Ra_t is plotted against dimensionless wave number a for different values of Brinkman-Darcy number ($\tilde{D}a$) as shown. This figure depicts that as Darcy number ($\tilde{D}a$) increases the thermal Rayleigh number Ra_t also increases. Therefore, Brinkman-Darcy number has stabilizing effect on the stationary convection which is in good agreement with the result obtained analytically in Eq. (42).

4. Conclusions

Thermal instability in a Darcy-Brinkman porous medium layer saturated by a Rivlin-Ericksen viscoelastic fluid layer heated from below in electrohydrodynamics has been investigated for the case of free-free boundaries by using perturbation theory and linear stability analysis. For the case of stationary convection, the non-Newtonian electrohydrodynamic Rivlin-Ericksen viscoelastic fluid behaves like an ordinary Newtonian fluid. AC electric field and Darcy number both have destabilizing influence while Brinkman-Darcy number has stabilizing influence on the onset of stationary convection.

List of Symbols

q	Velocity vector
a	Wave number
d	Thickness of the horizontal layer
E	Root-mean-square value of the electric field
$\mathbf{E}_0$	Root-mean-square value of the electric field at z = 0
g	Acceleration due to gravity
k_1	Medium permeability
K	Dielectric constant
K_0	Reference dielectric constant at T_0
l, m	Wave numbers in x and y directions
P	Modified pressure, defined by Eq. 4
Pr	Prandtl number, defined by Eq. 26a
F	Viscoelasticity parameter, defined by Eq. 26b
Da	Darcy number, defined by Eq. 26c
$\widetilde{Da}$	Brinkman-Darcy number, defined by Eq. 26d
Ra_t	Thermal Rayleigh number, defined by Eq. 27
Ra_e	AC electric Rayleigh number, defined by Eq. 28
t	Time
T	Temperature
T_0	Temperature at the lower boundary
T_1	Temperature at the upper boundary
V	Root-mean-square value of the electric potential
W	Amplitude of vertical component of perturbed velocity
k	Thermal conductivity
(x,y,z)	space co-ordinates

Greek symbols

μ Viscosity of fluid

μ' Viscoelastisity of fluid

α Coefficient of thermal expansion

γ Coefficient of thermal expansion of dielectric constant

κ Thermal diffusivity of the fluid

ϕ Medium porosity

ρ Density of fluid

ρ_e Free charge density

σ Electrical conductivity of fluid

ω Growth rate of disturbances

$\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ Horizontal Laplacian operator

$\nabla = \nabla_h^2 + \partial^2/\partial y^2$ Laplacian operator

Φ Amplitude of perturbed dielectric potential V

Θ Amplitude of perturbed temperature T

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Variable permeability and Soret effect on MHD radiative and reacting flow of viscoelastic fluid past an infinite porous plate in slip flow regime

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Abstract

The effect of heat and mass transfer on free convective flow of a visco-elastic incompressible and electrically conducting fluid past a vertical porous plate through a porous medium with time dependent oscillatory permeability and suction in presence of a uniform transverse magnetic field, radiation, chemical reaction and Soret effect in slip flow regime have been analysed.The coupled nonlinear partial differential equations are turned to ordinary by super imposing a solution with steady and time dependent transient component. Numerical value of velocity, temperature, skin friction, Nusselt number and Sherwood number for different value of the parameters involved in the problem are expressed through the graphs and table and discussed.

Keywords: MHD, Viscoelastic, Radiation, Soret effect, Variable permeability, Suction, Slip flow regime.

Introduction

An important study of two dimensional time dependent flow problem dealing with the response of boundary layer to external unsteady fluctuations of the free stream velocity about a mean value attracted the attention of many researchers Mishra *et al.* [10]. MHD flow with heat and mass transfer has been a subject of interest of many researchers because of its varied application in science and technology. Such phenomena are observed in buoyancy induced motions in the atmosphere, in water bodies, quasi-solid bodies such as earth, etc. In natural processes and industrial applications many transportation processes exist where transfer of heat and mass takes place simultaneously as a result of thermal diffusion and diffusion of chemical species. Several researchers have analyzed the free convective and mass transfer flow of a viscous fluid through porous medium. The permeability of the porous medium is assumed to be constant while the porosity of the medium may not be necessarily constant. Kim [8] studied the unsteady MHD convective heat past a semi infinite vertical porous moving plate with variable suction. The problem of three dimensional free convective flow and heat transfer through porous medium with periodic permeability has been discussed by Singh and Sharma [19]. Singh and Singh [18] have analyzed the heat and mass transfer in MHD flow of a viscous fluid past a vertical plate under oscillatory suction velocity. The study of viscoelastic fluids through porous medium has become the basis of many scientific and engineering applications. This type of flow is of great importance in the petroleum engineering concerned with the movement of oil, gas and water through reserviour of oil and gas field and to the hydrologist in the study of the migration of underground water, to the chemical engineers for the purification and filteration process and in the case of drug permeation through human skin.The principle of this subject are very useful in recovering the water for drinking and irrigation purpose. Gorla *et al*. [5] studied mixed convection effect on melting from a vertical plate in a porous medium. Narayana and Sibanda [13] presented influence of the Soret effect and double dispersion on MHD mixed convection along a vertical flat plate in non-darcy porous medium. Unsteady MHD flow of a visco-elastic fluid along vertical porous surface with chemical reaction was investigated by Nayak *et al.* [14]. Jha and Choudhary [7] studied influence of Soret effect on MHD mixed convection flow of viscoelastic fluid past a vertical surface with Hall current. Radiative convective flows are frequently encountered in many scientific and enviromental process, such as astrophysical flows, water evaporation from open reservoirs, heating and cooling of chambers and solar power technology. Several researchers have investigated radiative effects on heat transfer in non porous and porous medium utilizing the radiative heat flux model. Garg [6] studied magneto hydrodynamics and radiation effects on the flow due to moving vertical porous plate with variable temperature. Effects of chemical reaction and radiation on an unsteady MHD flow past an accelerated infinite vertical plate with variable temperature and mass transfer is presented by Ahmed *et al.* [1]. Rana [15] studied free convection effects on the oscillatory flow past a vertical porous plate in the presence of radiation for an optically thin fluid. Lavanya and Kesavaiah [9] presented radiation and Soret effects to MHD flow in vertical surface with chemical reaction and heat generation through a porous medium. Reddy [16] investigated unsteady heat and mass transfer MHD flow of a chemically reacting fluid past an impulsively started vertical plate with radiation. Visco-elastic MHD free convective flow through porous media in presence of radiation and chemical reaction with heat and mass transfer analysed by Choudhury and Das [2]. The problem of slip flow regime is very important in the era of modern science, technology and vast ranging industrialization. Rao *et al.* [17] analysed MHD transient free convection and chemically reactive flow past a porous vertical plate with radiation and temperature gradient dependent heat source in slip flow regime. Mukhopadhyay *et al.* [11] studied MHD mixed convection slip flow and heat transfer over a vertical porous plate. The objective of the present study is to analyze the variable permeability and Sorret effect on MHD radiative and reacting flow of viscoelastic fluid past an infinite porous plate in slip flow regime.

Mathematical formulation and analysis

The unsteady free convective flow of a visco-elastic (Walters B′) fluid past an infinite vertical porous plate in a porous medium with time dependent oscillatory suction as well as permeability in presence of a transverse magnetic field is considered. Let $x' - axis$ is assumed to be oriented vertically upwards along the plate and $y' - axis$ is taken normal to the plane of the plate. It is assumed that plate is electrically non-conducting and a uniform magnetic field of strength B_0 is applied normal to the plane of the plate.

The plate is subjected to a variable suction

$$v' = -V_0\left(1 + \varepsilon e^{i\omega' t'}\right) \tag{1}$$

and the permeability of the porous medium is asssumed to be of the form

$$k' = k_p'\left(1 + \varepsilon e^{i\omega' t'}\right) \tag{2}$$

Under usual Boussinesq's approximation, the governing equations and boundary conditions relevant to the physical model is given by

Equation of continuity

$$\frac{\partial v'}{\partial y'} = 0 \tag{3}$$

Equation of motion

$$\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} - k_0\frac{\partial^3 u'}{\partial t'\partial y'^2} + g\beta(T' - T_\infty') + g\beta_c(C' - C_\infty') - \frac{\nu}{k'}u' - \frac{\sigma B_0^2}{\rho}u' \tag{4}$$

Equation of energy

$$\frac{\partial T'}{\partial t'} + v'\frac{\partial T'}{\partial y'} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T'}{\partial y'^2} - \frac{1}{\rho c_p}\frac{\partial q'}{\partial y'} \tag{5}$$

Equation of mass transfer

$$\frac{\partial C'}{\partial t'} + v'\frac{\partial C'}{\partial y'} = D\frac{\partial^2 C'}{\partial y'^2} - K_1(C' - C_\infty') + D_1\frac{\partial^2 T'}{\partial y'^2} \tag{6}$$

Boundary conditions relevant to problem are:

$$\left.\begin{aligned} y' = 0, u' = L'\frac{\partial u'}{\partial y'}, T' = T_w' + \varepsilon(T_w' - T_\infty')e^{i\omega' t'}, C' = C_w' + \varepsilon(C_w' - C_\infty')e^{i\omega' t'} \\ y' \to \infty, u' \to 0, T' \to T_\infty', C' \to C_\infty' \end{aligned}\right\} \tag{7}$$

Where $L^{'} = \left(\frac{2-f_1}{f_1}\right)L$, with f_1 Maxwell reflexion coefficient $L = \mu\left(\frac{\pi}{2p\rho}\right)^{\frac{1}{2}}$ is mean free path and is a constant for an incompressible fluid, $L^{'}$ is the characterstics length of the plate.

Following Cogley *et al.* [3] the radiative heat flux is taken be of the form

$$\frac{\partial q^{'}}{\partial y^{'}} = 4\alpha^{'}(T^{'} - T_{\infty}^{'}) \tag{8}$$

Outside the boundary layer, the pressure term is assumed to be constant i.e.

$$-\frac{1}{\rho}\frac{\partial p^{'}}{\partial x^{'}} = 0 \tag{9}$$

Using equation (9), then equation (4) becomes,

$$\frac{\partial u^{'}}{\partial t^{'}} + v^{'}\frac{\partial u^{'}}{\partial y^{'}} = \nu\frac{\partial^2 u^{'}}{\partial y^{'2}} - k_0\frac{\partial^3 u^{'}}{\partial t^{'}\partial y^{'2}} + g\beta(T^{'} - T_{\infty}^{'}) + g\beta_c(C^{'} - C_{\infty}^{'}) - \frac{\nu}{k^{'}}u^{'} - \frac{\sigma B_0^2}{\rho}u^{'} \tag{10}$$

Introducing the following non-dimensional quantities,

$$\left.\begin{array}{c} y = \frac{V_0 y^{'}}{\nu}, t = \frac{V_0^2 t^{'}}{4\nu}, u = \frac{u^{'}}{V_0}, \theta = \frac{T^{'}-T_{\infty}^{'}}{T_w^{'}-T_{\infty}^{'}}, G_r = \frac{g\beta(T_w^{'}-T_{\infty}^{'})}{V_0^3}, \\ N = \frac{2\alpha^{'}\nu}{V_0\sqrt{\kappa}}, M = \frac{\sigma\nu B_0^2}{\rho V_0^2}, P_r = \frac{\mu c_p}{\kappa}, \alpha = \frac{k_0 V_0^2}{\nu^2}, k_p = \frac{k_p^{'} V_0^2}{\nu^2}, \omega = \frac{4\nu\omega^{'}}{V_0^2} \\ h = \frac{V_0 L^{'}}{\nu}, S_c = \frac{\nu}{D}, K_1 = \frac{K_1^{'}\nu}{D}, S_r = \frac{D_1(T_w^{'}-T_{\infty}^{'})}{\nu(C_w^{'}-C_{\infty}^{'})}, G_c = \frac{g\beta_c(C_w^{'}-C_{\infty}^{'})}{V_0^3} \end{array}\right\} \tag{11}$$

Using above non-dimensional quantities in equations (5), (6) and (9) the governing equations in non dimensional form,

$$\frac{1}{4}\frac{\partial u}{\partial t} - \left(1 + \varepsilon e^{i\omega t}\right)\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{1}{4}\alpha\left(\frac{\partial^3 u}{\partial t\partial y^2}\right) + G_r\theta + G_c C - \frac{1}{k_p(1+\varepsilon e^{i\omega t})}u - Mu \tag{12}$$

$$\frac{1}{4}P_r\frac{\partial\theta}{\partial t} - \left(1 + \varepsilon e^{i\omega t}\right)P_r\frac{\partial\theta}{\partial y} = \frac{\partial^2\theta}{\partial y^2} - N^2\theta \tag{13}$$

$$\frac{1}{4}\frac{\partial C}{\partial t} - \left(1 + \varepsilon e^{i\omega t}\right)\frac{\partial C}{\partial y} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} - K_1 C + S_r\frac{\partial^2\theta}{\partial y^2} \tag{14}$$

The non-dimensional boundary conditions are

$$\left.\begin{array}{c} y = 0, u = h\frac{\partial u}{\partial y}, \theta = 1 + \varepsilon e^{i\omega t}, C = 1 + \varepsilon e^{i\omega t} \\ y \to \infty, u \to 0, \theta \to 0, C \to 0 \end{array}\right\} \tag{15}$$

Method of solution

In view of periodic suction and permeability at the plate, following Das et al. [4] and Mishra *et al.* [12] and the velocity temperature concentration in the neighbourhood of the plate is assumed to be of the form:

$$\begin{Bmatrix} u(y.t) = u_0(y) + \varepsilon e^{i\omega t} u_1(y) \\ \theta(y.t) = \theta_0(y) + \varepsilon e^{i\omega t} \theta_1(y) \\ C(y.t) = C_0(y) + \varepsilon e^{i\omega t} C_1(y) \end{Bmatrix} \tag{16}$$

Using equation (16) in equations (12) to (14) and equating the harmonic and non-harmonic terms on both sides of equations, we get the following set of equations,

$$u_0'' + u_0' - \left(M + \frac{1}{k_p}\right) u_0 = -G_r\theta_0 - G_c C_0 \tag{17}$$

$$\left(1 - \frac{\alpha(i\omega)}{4}\right) u_1'' + u_1' - \left(M + \frac{1}{k_p} + i\omega\right) u_1 = -2u_0' - u_0'' + Mu_0 - G_r\theta_0 - G_c C_0 - G_r\theta_1 - G_c C_1 \tag{18}$$

$$\theta_0'' + P_r\theta_0' - N^2\theta_0 = 0 \tag{19}$$

$$\theta_1'' + P_r\theta_1' - \left[N^2 + \left(\frac{i\omega}{4}\right) P_r\right] \theta_1 = -P_r\theta_0' \tag{20}$$

$$C_0'' + S_c C_0' - K_1 C_0 = -S_c S_r \theta_0'' \tag{21}$$

$$C_1'' + S_c C_1' - S_c \left[K_1 + \left(\frac{i\omega}{4}\right)\right] C_1 = -S_c S_r \theta_1'' - S_c C_0' \tag{22}$$

The transform boundary conditions are,

$$\left.\begin{matrix} y = 0, u_0 = h\frac{\partial u_0}{\partial y}, u_1 = h\frac{\partial u_1}{\partial y}, \theta_0 = 1, \theta_1 = 1, C_0 = 1, C_1 = 1 \\ y \to \infty, u_0 \to 0, u_1 \to 0, \theta_0 \to 0, \theta_1 \to 0, C_0 \to 0, C_1 \to 0 \end{matrix}\right\} \tag{23}$$

Solving differential equations (17) to (22) under the boundary conditions (23), we get the following expression for velocity, temperature and concentration profile,

$$u(y,t) = \begin{Bmatrix} X_8 e^{-A_5 y} + X_6 e^{-A_1 y} - X_7 e^{-A_3 y} \\ +\varepsilon e^{i\omega t} \begin{bmatrix} X_9 e^{-A_1 y} + X_{10} e^{-A_2 y} + X_{11} e^{-A_3 y} \\ -X_{12} e^{-A_4 y} + X_{13} e^{-A_5 y} + X_{14} e^{-A_6 y} \end{bmatrix} \end{Bmatrix} \tag{24}$$

$$\theta(y,t) = e^{-A_1 y} + \varepsilon e^{i\omega t}\left[(1 - X_1) e^{-A_2 y} + X_1 e^{-A_1 y}\right] \tag{25}$$

$$C(y,t) = \begin{Bmatrix} (1 + X_2) e^{-A_3 y} - X_2 e^{-A_1 y} \\ +\varepsilon e^{i\omega t} \begin{bmatrix} -X_5 e^{-A_1 y} - X_3 e^{-A_2 y} + X_4 e^{-A_3 y} \\ +(1 + X_3 - X_4 + X_5) e^{-A_4 y} \end{bmatrix} \end{Bmatrix} \tag{26}$$

Some important characteristics of the flow field

From the velocity profile the skin friction at the plate in terms of amplitude and phase angle in non dimensional form is given by,

$$\tau = \left(\frac{\partial u_0}{\partial y}\right)_{y=0} + \varepsilon e^{i\omega t}\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = \left(\frac{\partial u_0}{\partial y}\right)_{y=0} + \varepsilon|D|cos(\omega t + \psi) \qquad (27)$$

where ,$|D| = \sqrt{D_r^2 + D_i^2},\ \psi = tan^{-1}\left(\frac{D_i}{D_r}\right)$

From the velocity profile the rate of heat transfer in terms of amplitude and phase in non-dimensional form is given by,

$$N_u = \left(\frac{\partial \theta_0}{\partial y}\right)_{y=0} + \varepsilon e^{i\omega t}\left(\frac{\partial \theta_1}{\partial y}\right)_{y=0} = \left(\frac{\partial \theta_0}{\partial y}\right)_{y=0} + \varepsilon|G|cos(\omega t + \gamma) \qquad (28)$$

where, $|G| = \sqrt{G_r^2 + G_i^2},\ \gamma = tan^{-1}\left(\frac{G_i}{G_r}\right)$

From the velocity profile the mass transfer coefficient, i.e, the Sherwood number at the plate in terms of amplitude and phase in non-dimensional form is given by,

$$s_h = \left(\frac{\partial C_0}{\partial y}\right)_{y=0} + \varepsilon e^{i\omega t}\left(\frac{\partial C_1}{\partial y}\right)_{y=0} = \left(\frac{\partial C_0}{\partial y}\right)_{y=0} + \varepsilon|H|cos(\omega t + \delta) \qquad (29)$$

where, $|H| = \sqrt{H_r^2 + H_i^2},\ \delta = tan^{-1}\left(\frac{H_i}{H_r}\right)$

Results and discussion

The problem of MHD free convection flow under the effect of thermal radiation and Soret number through porous medium with infinite vertical porous plate in the presence of chemical reaction is analysed.The closed form solutions for the velocity, temperature and concentration profile are obtained analytically and then evaluated numerically for different value of governing parameters. To have better insight of physical problem the variations of physical quantities with flow parameters are shown graphically. To be realistic the value of Prandtl number (P_r) are chosen to be 0.71 and 7 which correspond to air and water respectively. The values of Schmidt number (S_c) are chosen to represent hydrogen $(S_c = 0.66)$.The value of Grashoff number $(G_r > 0)$ are taken for cooling the plates. The values of Soret number, Hartmann number and radiation parameter are chosen arbitrary with $\varepsilon = 0.0001,\ \omega t = \frac{\pi}{2}$. It is clear from figure 1 that Grashoff number (G_r) and modified Grashoff number(G_c) enhance the fluid velocity. This figure also reveals that fluid velocity reduces with the increase of Hartmann number (M) and Prandtl number(P_r). From figure 2 we observed that fluid velocity increases with slip parameter (h), Soret number(S_r), porosity parameter $\left(k_p\right)$ and dimnishes with the increase of visco-elastic parameter (α). Figure 3 illustrate that fluid temperature decreases with the increase of Prandtl number (P_r) and radiation parameter (N). It is clear from figure 4 that with increase in Schmidt number and chemical reaction parameter the concentration profile decreases. Figure 4 also illustrate that fluid concentration increases with increase in Soret number (S_r). Table 1 present the variations in skin friction coefficient (τ), its amplitude and phase angle with $\omega t = \frac{\pi}{2}$ and $\varepsilon = 0.0001$. It observed from this table that an increase in Grashoff number modified Grashoff number, permeability of porous medium and Soret number lead to an increase in the value of amplitude and coefficient of skin friction, while an increase in Hartmann number, viscoelastic parameter, slip parameter and Prandtl number leads to decrease the amplitude and coefficient of skin friction. The value of phase angle increases due to increase in Hartmann number and Grashoff number, while decreases with increase in modified Grashoff number, viscoelastic parameter, permeability of porous medium, slip parameter, Soret number and Prandtl number. The variation in Nusselt number, its amplitude and phase angle with $\omega t = \frac{\pi}{2}$ and $\varepsilon = 0.0001$ is listed in Table 2. It is noticed from this table that with the increase in radiation parameter and frequency of oscillation amplitude of Nusselt number increases. The rate of heat transfer and phase angle is small but amplitude is very large in case of water $(P_r = 7.0)$ than in case of air $(P_r = 7.0)$. The values in the table clearly show that rate of heat transfer decreases and phase angle increase with the increase in radiation parameter. It is interesting to note that the amplitude, phase angle and rate of heat transfer all decreases with the increase in the frequency of oscillation. The numerical values of Sherwood number, its amplitude and phase angle with $\omega t = \frac{\pi}{2}$ and $\varepsilon = 0.0001$ are listed in table 3. From the table it is clear that amplitude and Sherwood number enhances with increase of Schmidt number, chemical reaction parameter and Soret number, while phase angle decreases with increase in these parameters.

Conclusion

The main conclusion of this study is:

1. The Soret number and permeability of porous medium enhance the fluid velocity.
2. Slip parameter has a tendency to increase the velocity of fluid.
3. Viscoelastic parameter reduces the fluid velocity. This is due to the fact that, elastic property in visco-elastic fluid reduces the frictional drag.
4. Hartmann number retards the flow due to the magnetic pull of Lorentz force.
5. Combined effect of increasing values of Prandtl number and radiation parameter is to reduce the temperature.
6. Skin friction is diminished with enhancement in Hartmann number.

7. The rate of heat transfer is more in case of air than in water.

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M	G_r	G_c	α	k_p	h	S_r	P_r	$\|D\|$	ψ	τ
2	5	5	0.05	0.5	0.2	0.8	0.71	3.9029	-0.17202	3.7669
4	5	5	0.05	0.5	0.2	0.8	0.71	2.8338	-0.12719	2.7708
2	10	5	0.05	0.5	0.2	0.8	0.71	5.1071	-0.16424	5.0442
2	5	10	0.05	0.5	0.2	0.8	0.71	6.6018	-0.17804	6.2566
2	5	5	0.2	0.5	0.2	0.8	0.71	3.847	-0.20115	3.8675
2	5	5	0.05	1	0.2	0.8	0.71	4.1609	-0.20839	4.5421
2	5	5	0.05	0.5	0.4	0.8	0.71	2.8928	-0.18732	2.8534
2	5	5	0.05	0.5	0.2	1.6	0.71	4.1745	-0.17575	3.9805
2	5	5	0.05	0.5	0.2	0.8	7	2.9602	-0.17392	2.9935

Table 1. Variation of skin friction(τ), its amplitude$|D|$ and phase angle.

N	P_r	ω	$\|G\|$	γ	N_u
1	0.71	5	1.9559	.00076213	-1.4176
4	0.71	5	4.7596	.00078657	-4.3745
1	7	5	13.865	.00071537	-7.1500
1	0.71	10	1.1027	.00043080	-1.517

Table 2. Variations in Nusselt number (N_u), its amplitude |G| and phase angle.

S_c	K_1	S_r	$\|H\|$	δ	S_h
0.22	0.02	0.8	0.13657	0.67528	-0.7779
0.66	0.02	0.8	0.28089	0.19153	0.004703
0.22	0.08	0.8	0.16496	0.52523	-0.12632
0.22	0.02	1.6	0.24225	-0.3742	0.34218

Table 3. Variations in Sherwood number(S_h), its amplitude |H| and phase angle.

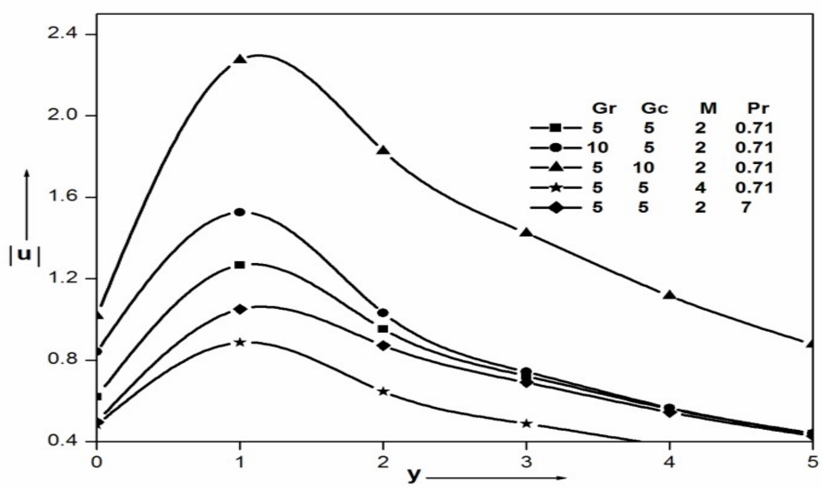

Fig.1. Variation of amplitude of velocity.

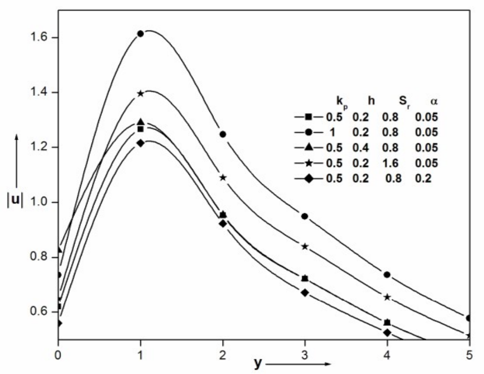

Fig.2. Variation of amplitude of velocity.

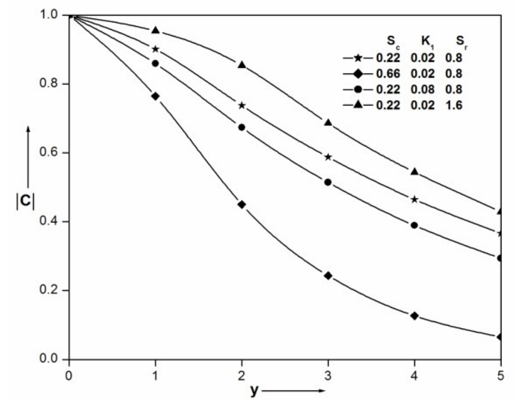

Fig.4. Variation of amplitude of concentration.

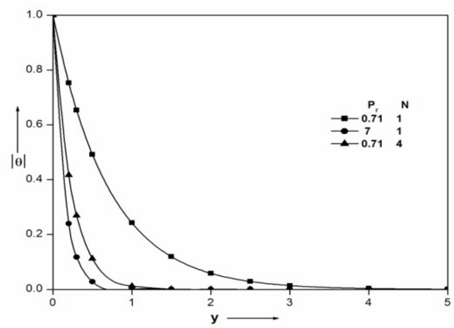

Fig.3. Variation of amplitude of temperature.

Appendix

$$A_1 = \frac{P_r + \sqrt{P_r^2 + 4N^2}}{2},\ A_2 = \frac{P_r + \sqrt{P_r^2 + 4\left(N^2 + \frac{(i\omega)P_r}{4}\right)}}{2},\ A_3 = \frac{S_c + \sqrt{S_c^2 + 4S_cK_1}}{2},$$

$$A_4 = \frac{S_c + \sqrt{S_c^2 + 4S_c\left(K_1 + \frac{i\omega}{4}\right)}}{2},\ A_5 = \frac{1 + \sqrt{1 + 4\left(M + \frac{1}{k_p}\right)}}{2},\ A_6 = \frac{1 + \sqrt{1 + 4\left(1 - \frac{\alpha(i\omega)}{4}\right)\left(M + \frac{1}{k_p} + i\omega\right)}}{2}$$

$$X_1 = \frac{A_1P_r}{A_1^2 - A_1P_r - \left[N^2 + \frac{(i\omega)P_r}{4}\right]},\ X_2 = \frac{S_rS_cA_1^2}{A_1^2 - A_1S_c - S_cK_1},\ X_3 = \frac{S_rS_cA_2^2(1 - X_1)}{A_2^2 - A_2S_c - S_c\left(K_1 + \frac{i\omega}{4}\right)}$$

$$X_4 = \frac{S_cA_3(1 + X_2)}{A_3^2 - A_3S_c - S_c\left(K_1 + \frac{i\omega}{4}\right)},\ X_5 = \frac{S_cA_1(A_1S_rX_1 + X_2)}{A_1^2 - A_1S_c - S_c\left(K_1 + \frac{i\omega}{4}\right)},\ X_6 = \frac{X_2G_c - G_r}{A_1^2 - A_1 - \left(M + \frac{1}{k_p}\right)},$$

A SUFFICIENT CONDITION FOR THE VALIDITY OF THE EXCHANGE PRINCIPLE IN TRIPLY DIFFUSIVE CONVECTION IN POROUS MEDIUM

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Abstract: Condition for characterizing nonoscillatory motions, which may be neutral or unstable, for triply diffusive convection in a porous medium is derived. It is analytically proved that the principle of the exchange of stabilities, in triply diffusive convection in a porous medium, is valid in the regime $\frac{R_1 E_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2 \pi^4} \leq 1$, where R_1 and R_2 are the concentration Raleigh numbers, and τ_1 and τ_2 are the Lewis numbers for the two concentration components respectively, σ is the Prandtl number, E_1 and E_2 are constants. It is further proved that this result is uniformly valid for all combinations of rigid and dynamically free boundaries.

Keywords: Triply diffusive convection, Porous medium, Darcy-Brinkman model, The principle of the exchange of stabilities, Concentration Rayleigh number.

1. INTRODUCTION

Research on convective fluid motion in porous media under the simultaneous action of a uniform vertical temperature gradient and a gravitationally opposite uniform vertical concentration gradient (known as double diffusive convection) has been an area of great activity due to its importance in the predication of ground water movement in aquifers, in assessing the effectiveness of fibrous materials, in engineering geology and in nuclear engineering. Double diffusive convection is now well known. For a broad view of the subject one may be referred to Nield and Bezan (2006), Murray and Chen (1989), Nield (1968), Taunton et al. (1972), Kuznetsov and Nield (2008), Lombardo and Mulone (2002), Basu and Layek (2013).

All these researchers have considered double diffusive convection. However, it has been recognized later that there are many fluid systems, in which more than two components are present. For example, Degens et al (1973) reported that the saline waters of geothermally heated Lake kivu are strongly stratified by heat and a salinity which is the sum of comparable concentrations of many salts. Similarly the oceans contain many salts having concentrations less than a few percent of the sodium chloride concentration. Multi-component concentrations can also be found in magmas and substratum of water reservoirs. The subject with more than two components (in porous and non porous medium) has attached the attention of many researchers Grifiths (1979a, 1979b), Poulikakos (1985), Pearlstein et al. (1989), Terrones and Pearlstein (1989), Rudraiah and Vortmeyer (1982), Lopez et al (1990), Tracey (1996, 1998), Rionero (2010), Straughan and Tracey (1999). The essence of the works of these researchers is that small salinity of a third component with a smaller mass diffusivity can have a significant effect upon the nature of convection; and 'oscillatory' and direct 'salt finger' modes are simultaneous possible under a wide range of conditions, when the density gradients due to components with greatest and smallest diffusivity are of same signs. Terrones (1993) studied the effects of cross-diffusion on the onset of convective instability in a horizontally infinite triply diffusive and triply stratified fluid layer. Ryzhkov and Shevtsova (2009) investigated the long-wave instability of a vertical multicomponent fluid layer induced by the Soret effect. Rionero (2013a) investigated a triply convective diffusive fluid mixture saturating a porous layer and derived sufficient conditions for inhibiting the onset of convection. Rionero (2013b) further studied the multicomponent diffusive convection in porous layer salted by m salts partly from above and partly from below.

The validity of the principle of the exchange of stabilities (PES) (i.e. nonoccurence of oscillatory motions) in stability problems removes the unsteady terms from the linear perturbation equations which results in notable mathematical simplicity since the transition from stability to instability occurs via a marginal state which is defined by the vanishing of both real and imaginary parts of the complex time eigenvalue associated with the perturbation. Pellew and southwell (1940) proved the validity of PES for Rayleigh-Benard problem. However no such result exists for other more complex hydrodynamic configurations. Banerjee et al (1985) derived a sufficient condition for the validity of PES for hydromagnetic Rayleigh-Benard problem. Gupta et al (1986) extended Banerjee et al's (1985) criterion to rotatory hydromagnetic thermohaline convection problem. To the author's knowledge no such result exists for triply diffusive convection in porous medium. Thus the present paper which provides a sufficient condition for the validity of PES in triply diffusive convection in porous medium may be regarded as a first step in this scheme of extended investigations. The following result is obtained in this direction:

For triply diffusive convection in porous medium, if $\frac{R_1 E_1 \sigma}{2\tau_1^2\pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2\pi^4} \leq 1$, then an arbitrary neutral or unstable mode of system is definitely nonoscillatory in character and in particular PES is valid where R_1 and R_2 are the concentration Raleigh numbers, and τ_1 and τ_2 are the Lewis numbers for two concentration components respectively, σ is the Prandtl number, E_1 and E_2 are constants. It is further proved that this result is uniformly valid for all combinations of rigid and dynamically free boundaries and the results for Rayleigh-Benard convection in porous medium and double diffusive convection in porous medium follow as a consequence

2. MATHEMATICAL FORMULATION AND ANALYSIS

A viscous finitely heat conducting Boussinesq fluid layer, saturating a porous medium, of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and $T_1(< T_0)$ and uniform concentrations S_{10}, S_{20} and $S_{11}(< S_{10}), S_{21}(< S_{20})$(as shown in Fig.1). It is assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Brinkman extended Darcy model has been used to investigate the triple diffusive convection in porous medium.

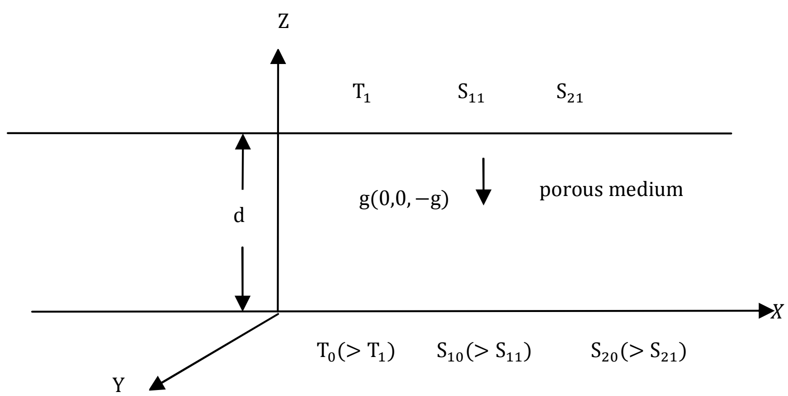

Fig.1

The governing equations of triply diffusive convection in porous medium (Darcy-Brinkman model), in the non-dimensional form are given by (Vafai (2006))

$$\Lambda(D^2 - a^2)^2 w - (p + D_a^{-1})(D^2 - a^2)w = R\, a^2\theta - R_1 a^2 \phi_1 - R_2 a^2 \phi_2\, , \tag{1}$$

$$(D^2 - a^2 - E\,\sigma p)\theta = -w\, , \tag{2}$$

$$\left(D^2 - a^2 - \frac{E_1 \sigma\, p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}\, , \tag{3}$$

$$\left(D^2 - a^2 - \frac{E_2 \sigma\, p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}\, . \tag{4}$$

The equations (1) – (4) are to be solved by using the following boundary conditions:

$w = \theta = \phi_1 = \phi_2 = Dw = 0$ at $z = 0$ and at $z = 1$,(when both the boundaries are rigid) (5) or $w = \theta = \phi_1 = \phi_2 = D^2w = 0$ at $z = 0$ and at $z = 1$,(when both the boundaries are free) (6)

$$\left.\begin{array}{l}\text{or}\quad w = \theta = \phi_1 = \phi_2 = Dw = 0 \text{ at } z = 0, \ \text{(when lower boundary is rigid)} \\ \text{and}\quad w = \theta = \phi_1 = \phi_2 = D^2w = 0 \text{ at } z = 1, \ \text{(when upper boundary is free)}\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l}\text{or}\quad w = \theta = \phi_1 = \phi_2 = D^2w = 0 \text{ at } z = 0, \ \text{(when lower boundary is free)} \\ \text{and}\quad w = \theta = \phi_1 = \phi_2 = Dw = 0 \text{ at } z = 1, \ \text{(when upper boundary is rigid)}\end{array}\right\} \tag{8}$$

where z is the real independent such that $0 \leq z \leq 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $\sigma > 0$ the Prandtl number, $\tau > 0$ is the Lewis number, $R > 0$ is the Rayleigh number, $R_1 > 0$ and $R_2 > 0$ are the two concentration Rayleigh numbers, $p = p_r + ip_i$ is the complex growth rate where p_r and p_i are the real constants, w is the vertical velocity, θ, is the temperature, ϕ_1 and ϕ_2 are the two concentrations. It may further be noted that in Eqs. (1)-(4) together with the boundary conditions (5) or (6) or (7) or (8) describe an eigenvalue problem for p and govern triply diffusive convection in porous medium for any combination of dynamically free and rigid boundaries.

Now we prove the following theorem

Theorem. If $(w, \theta, \phi_1, \phi_2, p)$, $p = p_r + ip_i$, $p_r \geq 0$ is a solution of Eqs. (1) – (8) with $R > 0$, $R_1 > 0$, $R_2 > 0$ and $\frac{R_1E_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2E_2\sigma}{2\tau_2^2\pi^4} \leq 1$ then $p_i = 0$.In particular $p_r = 0$ implies $p_i = 0$, if $\left(\frac{R_1E_1\sigma}{2\tau_1^2\pi^4} + \frac{R_2E_2\sigma}{2\tau_2^2\pi^4}\right) \leq 1$.

Proof: Multiplying equation (1) by w* (the superscript * henceforth denotes complex conjugation) on both sides and integrating over vertical range of z, we obtain

$$\Lambda \int_0^1 w^* (D^2 - a^2)^2 w\, dz - (p + D_a^{-1}) \int_0^1 w^* (D^2 - a^2) w\, dz =$$
$$R\, a^2 \int_0^1 w^* \theta\, dz - R_1 a^2 \int_0^1 w^* \phi_1\, dz - R_2 a^2 \int_0^1 w^* \phi_2\, dz. \tag{9}$$

Making use of Eqs. (2) – (4) and the fact that w (0) = 0 = w (1), we can write

$$R\, a^2 \int_0^1 w^* \theta\, dz = -Ra^2 \int_0^1 \theta\, (D^2 - a^2 - E\, \sigma\, p^*)\theta^* dz, \tag{10}$$

$$R_1 a^2 \int_0^1 w^* \phi_1 dz = -R_1 a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{E_1 \sigma\, p^*}{\tau_1}\right) \phi_1^* dz, \tag{11}$$

$$R_2 a^2 \int_0^1 w^* \phi_2 dz = -R_2 a^2 \tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{E_2 \sigma\, p^*}{\tau_2}\right) \phi_2^* dz. \tag{12}$$

Combining Eqs. (9) – (12), we obtain

$$\Lambda \int_0^1 w^* (D^2 - a^2)^2 w \, dz - (p + D_a^{-1}) \int_0^1 w^* (D^2 - a^2) w \, dz =$$
$$-Ra^2 \int_0^1 \theta \,(D^2 - a^2 - E \sigma p^*)\theta^* dz + R_1 a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{E_1 \sigma p^*}{\tau_1}\right) \phi_1^* dz +$$
$$R_2 a^2 \tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{E_2 \sigma p^*}{\tau_2}\right) \phi_2^* dz. \tag{13}$$

Integrating various terms of equation (13) by parts for an appropriate number of times and making use of either of the boundary conditions (5) – (8), it follows that

$$\Lambda \int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) \, dz + (p + D_a^{-1}) \int_0^1 (|Dw|^2 + a^2 |w|^2) \, dz =$$
$$Ra^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + E\sigma p^* |\theta|^2) \, dz - R_1 a^2 \tau_1 \int_0^1 \Big(|D\phi_1|^2 + a^2 |\phi_1|^2 +$$
$$\frac{E_1 \sigma p^*}{\tau_1} |\phi_1|^2\Big) dz - R_2 a^2 \tau_2 \int_0^1 \left(|D\phi_2|^2 + a^2 |\phi_2|^2 + \frac{E_2 \sigma p^*}{\tau_2} |\phi_2|^2\right) dz \tag{14}$$

Equating imaginary parts on both sides of equation (14) and cancelling $p_i (\neq 0)$ throughout, we have

$$\int_0^1 (|Dw|^2 + a^2 |w|^2) \, dz = - Ra^2 E\sigma \int_0^1 |\theta|^2 dz + R_1 a^2 E_1 \sigma \int_0^1 |\phi_1|^2 dz +$$
$$R_2 a^2 E_2 \sigma \int_0^1 |\phi_2|^2 dz \, . \tag{15}$$

Now, multiplying equation (3) by its complex conjugate and integrating the resulting equation for a suitable number of times and use the boundary condition on ϕ_1 namely, $\phi_1(0) = 0 = \phi_1(1)$, we obtain

$$\int_0^1 (|D^2 \phi_1|^2 + 2a^2 |D\phi_1|^2 + a^4 |\phi_1|^2) \, dz + \frac{2E_1 \sigma p_r}{\tau_1} \int_0^1 (|D\phi_1|^2 + a^2 |\phi_1|^2) dz +$$
$$\frac{E_1^2 \sigma^2 |p|^2}{\tau_1^2} \int_0^1 |\phi_1|^2 dz = \frac{1}{\tau_1^2} \int_0^1 |w|^2 dz. \tag{16}$$

Since $p_r \geq 0$, it follows from equation (16), that

$$2a^2 \int_0^1 |D\phi_1|^2 dz < \frac{1}{\tau_1^2} \int_0^1 |w|^2 dz, \tag{17}$$

Now, since ϕ_1, ϕ_2 and w satisfy the boundary conditions $\phi_1(0) = 0 = \phi_1(1)$, $\phi_2(0) = 0 = \phi_2(1)$ and $w(0) = 0 = w(1)$ respectively, we have by Rayleigh-Ritz inequality (Schultz (1973))

$$\int_0^1 |D\phi_1|^2 dz \geq \pi^2 \int_0^1 |\phi_1|^2 dz, \tag{18}$$

$$\int_0^1 |D\phi_2|^2 dz \geq \pi^2 \int_0^1 |\phi_2|^2 dz, \tag{19}$$

$$\int_0^1 |Dw|^2 dz \geq \pi^2 \int_0^1 |w|^2 dz. \tag{20}$$

Utilizing inequalities (18) and (20) in inequality (17), we get

$$a^2 \int_0^1 |\phi_1|^2 dz < \frac{1}{2\tau_1^2 \pi^4} \int_0^1 |Dw|^2 dz, \tag{21}$$

In the same manner, by using inequalities (19) and (20), we obtain from Eq. (4), that

$$a^2 \int_0^1 |\phi_2|^2 dz < \frac{1}{2\tau_2^2 \pi^4} \int_0^1 |Dw|^2 dz. \tag{22}$$

Utilizing inequalities (21) and (22) in Eq. (15), we obtain

$$\left[1 - \left(\frac{R_1 E_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2 \pi^4}\right)\right] \int_0^1 |Dw|^2 dz + a^2 \int_0^1 |w|^2 dz + Ra^2 E\sigma \int_0^1 |\theta|^2 dz < 0\ . \tag{23}$$

which, clearly implies that

$$\frac{R_1 E_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2 \pi^4} > 1. \tag{24}$$

Hence if $\frac{R_1 E_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2 \pi^4} \leq 1$, then we must have $p_i = 0$.

This proves the theorem.

The essential content of the theorem from the physical point of view is that for the problem of triply diffusive convection in porous medium, an arbitrary neutral or unstable mode of the system is definitely nonoscillatory in character and in particular the principle of the exchange of stabilities is valid if $\frac{R_1 E_1 \sigma}{2\tau_1^2 \pi^4} + \frac{R_2 E_2 \sigma}{2\tau_2^2 \pi^4} \leq 1$. Further this result is uniformly valid for any combination of rigid and / or free boundaries.

Special Cases: It follows from theorem1 that an arbitrary neutral or unstable mode is non oscillatory in character and in particular PES is valid for:

1. Rayleigh-Benard convection in porous medium ($R_1 = R_2 = 0$)
2. Thermohaline convection in porous medium ($R_2 = 0$) if $\frac{R_1 E_1 \sigma}{2{\tau_1}^2 \pi^4} \leq 1$.

3. CONCLUSION

Linear stability theory is used to derive a sufficient condition for the validity of ‘the PES’ in triply diffusive convection in porous medium. It is further proved that this result is uniformly valid for any combination of rigid and / or free boundaries.

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PRIMARY AND G- PRIMARY FUZZY IDEALS OF A SEMIRING

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ABSTRACT: In this paper, we study primary fuzzy ideals and the effect of group action on primary fuzzy ideals of a semiring R with finite group action on it and show that the results derived in [12] and [13] concerning these ideals is of wider generality.

Keywords: Semirings; Primary fuzzy ideals; G- primary fuzzy ideals.

1. INTRODUCTION

In [12], we generalized the primary ideals from commutative rings to non commutative rings by replacing the role of elements by ideals. This definition of primary ideals coincides with the definition of primary ideals given in [14] through their associated primes under the assumption that the ring is Noetherian. In [13] we consider the group action of a finite group G on a semiring R and define G-primary ideals of R. Then results proved for primary ideals in [12] are carried over to G-primary ideals in [13]. The theory of fuzzy sets developed by Lotfi A. Zadeh [15] some 40 years ago has useful and important applications. Since the pioneering paper of Zadeh, research on the theory of fuzzy sets has been growing steadily. Many mathematicians have applied the concept of fuzzy subsets to the theory of groups and rings in algebra [3-7].

In this paper, after introducing the notions for primary ideals in [12] and G-primary ideals in [13] of a non commutative semiring, we achieve their fuzzyfication by proving a theorem that characterizes primary fuzzy ideals in terms of primary ideals of non commutative semirings.

Since the presentation in this paper is general towards the non commutative case, the results we obtain here are also valid for the commutative case.

2. PRELIMINARIES

Throughout this paper, (R, +, .) represents a semiring. First we recall some definitions of the basic concepts of semirings that we need in sequel.

Definition 2.1. A semiring is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(i) $(R, +)$ is a commutative monoid with identity element 0.

(ii) $(R, .)$ is a monoid with identity element 1.

(iii) Multiplication distributes over addition from either side.

(iv) $0.r = 0 = r.0$ for all $r \in R$

(v) $1 \neq 0$.

Definition 2.2. A nonempty subset (ideal) A of a semiring R is subtractive if and only if $a \in A$ and $a + b \in A$ implies that $b \in A$

Example 2.3. The set 2N of all nonnegative even integers is a subtractive ideal of the semiring (N, +, ·), where N is a set of all non negative integers.

Definition 2.4. An ideal P of a semiring R is prime if and only if whenever $AB \subseteq P$, for ideals A and B of R, we must have either $A \subseteq P$ or $B \subseteq P$

Definition 2.5. A G-invariant ideal I of a semiring R is said to be G-maximal if $I \neq R$ and for any G-invariant ideal J of R, $I \subseteq J \subseteq R$ implies that either $J = I$ or $J = R$

Definition 2.6. A G-invariant ideal A of a semiring R is a G-prime ideal if and only if for G-invariant ideals A_1, A_2 of a semiring R, $A_1 A_2 \subseteq A$ implies that either $A_1 \subseteq A$ or $A_2 \subseteq A$.

Definition 2.7. Let X be a nonempty set. A function $\lambda : X \to [0,1]$ is called a fuzzy set of X.

Example2.8. Let Z be the set of integers and λ function from set Z into [0,1] defined by

$\lambda(x) = \dfrac{1}{x^2 + 2}, x \in Z$ Then λ is a fuzzy set of Z.

Definition 2.9. Let λ be a fuzzy set of X. Then for $t \in [0,1]$, the set $\lambda_t = \{x \in X \mid \lambda(x) \geq t\}$ is called a t-level set of λ.

Definition 2.10. Let I be a nonempty index set and let $(\lambda_t)_{i \in I}$ be a family of fuzzy sets of X. Then the union $\bigcup_{i \in I} \lambda_i$ and the intersection $\bigcap_{i \in I} \lambda_i$ of the family $(\lambda_t)_{i \in I}$ is defined by

$\bigcup_{i \in I} \lambda_i(x) = \sup\{\lambda_i(x) \mid i \in I\}$ and $\bigcap_{i \in I} \lambda_i(x) = \inf\{\lambda_i(x) \mid i \in I\}$ for all $x \in X$. Here sup and inf denote the suprimum and the infimum respectively.

Definition 2.11. Let $(R, +, .)$ be a semiring. If λ and μ are two fuzzy subsets of R, then the product $\lambda \circ \mu$ is defined by

$(\lambda \circ \mu)(z) = \sup\limits_{z=xy}\{\min(\lambda(x), \mu(y))\}$ for all $z \in R$. The sum of two fuzzy subsets is defined analogously.

The fuzzy ideals and prime fuzzy ideals are defined in semirings as follows:

Definition 2.12. Let S be a semi group (monoid) and λ a fuzzy subset of S then λ is called a fuzzy subsemigroup (fuzzy submonoid) of S if and only if $\lambda(xy) \geq \min(\lambda(x), \lambda(y))$.

Definition 2.13. Let R be a semiring. A fuzzy subset λ of R is called fuzzy right (left) ideal of R if

(i) $\lambda(x-y) \geq \min(\lambda(x), \lambda(y))$

(ii) $\lambda(xy) \geq \lambda(x), (\lambda(xy) \geq \lambda(y))$.

λ is said to be fuzzy ideal of R if it is both a left and a right fuzzy ideal of R .

Definition 2.14. A fuzzy ideal P of R is called a prime fuzzy ideal if either $P = \chi_R$ or P is a nonconstant function and for any two fuzzy ideals λ and μ in R, $(\lambda \circ \mu) \subseteq P$ implies that either $\lambda \subseteq P$ or $\mu \subseteq P$.

3. PRIMARY FUZZY IDEALS

We define the fuzzy analogue of a primary ideal in a semiring as follows:

Definition 3.1. Let R be a semiring and P a fuzzy ideal of R. Then P is said to be primary fuzzy ideal if either $P = \chi_R$ or P is a nonconstant function and for any two fuzzy ideals λ and μ in R, $(\lambda \circ \mu) \subseteq P$ implies that either $\lambda \subseteq P$ or $\mu \subseteq \sqrt{P}$.

Definition 3.2. Let λ be a fuzzy ideal of a semiring R. The fuzzy radical of λ, denoted by $\sqrt{\lambda}$, is defined by $\sqrt{\lambda} = \bigcap\{P \mid P \in \rho_\lambda\}$ where ρ_λ denotes the family of all prime fuzzy ideals P of R such that $\lambda \subseteq P$ and $\lambda_* \subseteq P_*$ where $\lambda_* = \{x \in R \mid \lambda(x) = \lambda(0)\}$ and $P_* = \{x \in R \mid P(x) = P(0)\}$.

Note that (i) $\rho_\lambda \neq \phi$ as $\chi_R \in \rho_\lambda$.

(ii) If $\lambda(0) = 1$, then every prime fuzzy ideal P containing λ is in ρ_λ .

The following theorem describes primary fuzzy ideals of a semiring R in terms of primary ideals of R.

Theorem 3.3. Let R be a semiring and P a nonconstant fuzzy ideal of R. Then P is a primary fuzzy ideal of R if and only if $P_r \in \{\pi, R\}$, for all $r \in [0,1]$, where $\pi(\neq R)$ is a primary ideal of R.

Proof. Let $I(R)$ consist of the empty set ϕ together with all the ideals of R. Since P is nonconstant, the decreasing function $\psi : [0,1] \to I(R)$ given by $\psi(r) = P_r$ takes on at least one nonempty set other than R.

In fact, we show that it takes on exactly two values $P_t \neq \phi$ and $P_u = R$, for all $r \in [0,1]$ where $t = P(0) > u = P(1)$. If this is not the case, then there exists $s \in [0,1]$ with $t > s$ such that $P_t \subset P_s \subset R$. Define two fuzzy ideals λ and μ of R by

$$\lambda_r = \begin{cases} P_s .... r \in [0,1] \\ R .... r = 0 \end{cases} \qquad \text{and} \qquad \mu_r = \begin{cases} P_r .... r \succ s \\ R .... r \leq s \end{cases} .$$

As in **(c.f. [10], Proposition 3.3),** $(\lambda \circ \mu) \subseteq P$ and λ is not contained in P, since $\lambda_t = P_s \not\subset P_t$. We here show that $\mu \not\subset \sqrt{P}$, which will contradict the fact that P is primary. Since $P_s \neq R$, there exist m, 0<m<s such that $P_m \neq R$.

For if $P_m = R$ for all 0<m<s, then for any $x \in R$, we have $P(x) = \sup\{i \in [0,1] \ | x \in P_i\} \geq \sup\{m \in [0,1] \ | 0 < m < s\} = s$, implying that $x \in P_s$. This contradicts the fact that $P_s \neq R$. Moreover $m < s$ implies that $P_s \subseteq P_m$ so that $P_m \neq \phi$ as $P_s \neq \phi$. Since $\phi \subset P_m \subset R$ and P_m is an ideal of R, therefore there exists a prime ideal $I(\neq R)$) of R such that $P_m \subseteq I$. Define a fuzzy ideal P′ of R by $P'_r = \begin{cases} I .... r > m \\ R .... r \leq m \end{cases}$. The characterization Theorem **(c.f. [10], Theorem 3.4)** for prime fuzzy ideals implies that P′ is a prime fuzzy ideal of R, since I is a prime ideal of R. Moreover $r > m$ implies that $P_r \subseteq P_m \subseteq I = P_r'$ and $r \leq m$ implies that $P_r' = R$, so that $P_r \subseteq P_r'$. Thus $P \subseteq P'$. Also since $P_m \neq \phi$, we have $P_* \subseteq P_m \subseteq I = P_*'$. Further $P_s' = I \neq R$, since $r > m$. Now the fact that P′ is a prime fuzzy ideal of R, $P \supseteq P'$ $P_*' \supseteq P_*$ and $P_s' \neq R$, coupled with the fact that for any family $(\lambda_i)_{i \in I}$ of fuzzy ideals of R, we have $(\bigcap \lambda_i)_r = \bigcap (\lambda_i)_r$, $r \in [0,1]$, it follows that $\left(\sqrt{P}\right)_s \neq R$. However $\mu_s = R$.

Therefore $\mu_s \not\subset \left(\sqrt{P}\right)$ and consequently $\mu \not\subset \left(\sqrt{P}\right)$. Thus there exist an ideal $\pi(\phi \subset \pi \subset R)$ with $P_r \in \{\pi, R\}$ for all $r \in [0,1]$. It remains to prove that π is primary. Let α and β be two ideals of R with $\alpha\beta \subseteq \pi$. Then the characteristic functions χ_α and χ_β satisfy $\chi_\alpha \circ \chi_\beta \subseteq \chi_{\alpha\beta} \subseteq \chi_\pi \subseteq P$, because $(\chi_\pi)_r = \begin{cases} \pi .... 0 < r \leq 1 \\ R .... r = 0 \end{cases}$.

Now the primary character of P yields that either $\chi_\alpha \subseteq P$ or $\chi_\beta \subseteq \sqrt{P}$. If $\chi_\alpha \subseteq P$, then $\alpha \subseteq \pi$. Suppose $\chi_\alpha \not\subset P$, then $\chi_\beta \subseteq \sqrt{P}$. Let $\pi^{/}$ be any prime ideal of R containing π. Since $P_r \in \{\pi, R\}$ for all $r \in [0,1]$, we have

$$P_r = \begin{cases} \pi .... r > n \\ R .... r \leq n \end{cases}, \text{ for some } n \in [0,1). \qquad (1)$$

Define a prime fuzzy ideal P′ of R by $P_r^{/} = \begin{cases} \pi^{/} .... r > n \\ R .... r \leq n \end{cases}$. (2)

Since $\pi \subseteq \pi^{/}$, it is clear from (1) and (2) that $P^{/} \supseteq P$ and $P_*^{/} \supseteq P_*$. Therefore $\chi_\beta \subseteq \sqrt{P}$ implies that $\chi_\beta \subseteq P'$. This gives $\beta = \left(\chi_\beta\right)_I \subseteq \left(P^{/}\right)_I = \pi^{/}$, which by virtue of the fact that $\pi^{/}$ is any prime ideal of R containing π yields that $\beta \subseteq \sqrt{\pi}$. This completes the proof that π is primary. Now for the converse, assume that $P_r \in \{\pi, R\}$ for all $r \in [0,1]$, where π is a primary ideal of R. Suppose that P is not primary. Then there exist fuzzy ideals λ and μ of R with $\lambda \circ \mu \subseteq P$, but $\lambda \not\subset P$ and $\mu \not\subset P$. The later statements imply that $\lambda_s \not\subset P_s$ and $\mu_t \not\subset \left(\sqrt{P}\right)$ for some $r, s \in [0,1]$. Since every ideal is contained in R, $\lambda_s \not\subset P_s$ implies that $P_s = \pi$. Further, $\mu_t \not\subset \left(\sqrt{P}\right)$ implies that there exists a prime fuzzy ideal $P^{/}$ of R with $P^{/} \supseteq P$ and $P_*^{/} \supseteq P_* = \pi$ such that $\mu_t \not\subset P^{/}{}_t$, this again implies that $P_t^{/} = P_*^{/}$. Thus there exists a prime ideal $P_*^{/}$ containing π such that $\mu_t \not\subset P_*^{/}$ Consequently, $\mu_t \not\subset \sqrt{\pi}$. Let $m = (s,t)$. Then $s \geq m$ and $t \geq m$ implies that $\lambda_m \subseteq \pi$ and $\mu_m \not\subset \sqrt{\pi}$. But $\lambda \circ \mu \subseteq \pi$ implies that $P(ab) \geq \min(\lambda(a), \mu(b))$. Thus for any $z = \sum x_i y_i$ in $\lambda_m \mu_m$ where x_i gives $\lambda_m \mu_m \subseteq \pi$, which contradicts the primary character of π. Hence P is primary.

We now, use the Characterization Theorem 3.3 to derive the fuzzy analogous of various results proved in **[12].**

Theorem 3.4. (i) Let $P_1, P_2, P_3, \ldots\ldots\ldots P_n$, be primary fuzzy ideals of a semiring R such that $\sqrt{P_i} = \mu$ (i=1, 2,…, n). Then $P = \bigcap_{i=1}^{n} P_i$ is primary and $\sqrt{P} = \mu$.

(ii) Let R and R′ be two semirings and $T : R \mapsto R^{/}$ is an onto homomorphism. Let λ be a fuzzy ideal of R such that both λ_* and $\sqrt{\lambda_*}$ are subtractive and $K \subseteq \lambda_*$ where $K = \{x \in R \mid x = a + b, T(a) = T(b)\}$. If λ is primary, then $T(\lambda)$) is primary. Moreover, if range λ is finite and $T(\lambda)$ is primary, then λ is primary.

Proof.(i) By the characterization of primary fuzzy ideals $P_1, P_2, P_3, \ldots\ldots\ldots P_n$,, there exist primary ideals $\pi_1, \pi_2, \pi_3, \ldots\ldots\ldots \pi_n$, of semiring R and $m_1, m_2, m_3, \ldots\ldots\ldots m_n \in [0,1)$ such that $(P_i)_r = \begin{cases} \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r > m_i \\ R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r \le m_i \end{cases}$. It is to be noted that $\sqrt{P_1} = \sqrt{P_2} = \sqrt{P_3}. = \ldots\ldots = \sqrt{P_n}$ implies that $m_1 = m_2 = m_3 = \ldots\ldots\ldots = m_n = m$.

For if $m_i < m_j$ for some $i \ne j$, then choose a prime ideal $\pi \supseteq \pi_i$ and define a prime fuzzy ideal Q by $Q_r = \begin{cases} \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r > m_i \\ R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r \le m_i \end{cases}$. Clearly, $Q \in \rho_{P_i}$. But for $m_i < r < m_j$, $Q_r = \pi$, $R = (P_j)_r \not\subset \pi$. The later implies that there exists $x \in R$ such that $Q(x) < P_j(x)$ and the former implies that $(\sqrt{P_i})(x) \le Q(x)$ so that $(\sqrt{P_i})(x) \le Q(x) < P_j(x) \le (\sqrt{P_j})(x)$, contradicting that $\sqrt{P_i} = \sqrt{P_j}$. Therefore we have $P_r = \begin{cases} \bigcap_{i=1}^{n} \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r > m \\ R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r \le m \end{cases}$. Also, using $(\sqrt{P_i})_* = \sqrt{(P_i)_*}$, we get $\sqrt{\pi_1} = \sqrt{\pi_2} = \sqrt{\pi_3}. = \ldots\ldots = \sqrt{\pi_n}$. Now, using **(c.f.[12],Theorem3.19(i)(a))** and consequently by **Theorem 3.3**, P is primary. Moreover, using **(c.f. [12], Proposition 3.9 (iv))** and the fact $\left(\bigcap_{i=1}^{n} P_i\right)_r = \bigcap_{i=1}^{n} (P_i)_r$ and $\left(\sqrt{\bigcap_{i=1}^{n} P_i}\right)_r = \sqrt{\bigcap_{i=1}^{n} (P_i)_r}$, we have $\sqrt{P} = \sqrt{\bigcap_{i=1}^{n} P_i} = \bigcap_{i=1}^{n} \sqrt{P_i} = \mu$.

(ii) Let $T : R \mapsto R^{/}{}'$ be an onto homomorphism and $\lambda : R \mapsto [0,1]$ be a fuzzy ideal of R. If λ is primary, then obviously by Theorem 3.3 range λ is finite. We first show that $(T(\lambda))_r = T(\lambda_r)$ for all $r \in [0,1]$] if range λ is finite. For this, let $y \in T(\lambda_r)$. Then there exists $x \in \lambda_r$ such that $T(x) = y$. Now, $x \in \lambda_r$ implies $\lambda(x) \geq r$ and therefore $(T(\lambda))(y) = \sup_{T(x)=y} \{\lambda(x)\} \geq r$ .Thus $y \in (T(\lambda))_r$. Hence $T(\lambda_r) \subseteq (T(\lambda))_r$. To show that $(T(\lambda))_r \subseteq T(\lambda_r)$, let $y \in (T(\lambda))_r$. Then $T(\lambda)(y) \geq r$, that is $\sup_{T(z)=y} \lambda(z) = y$. Since range of λ is finite, therefore λ possesses the sup property, that is $\sup_{x \in E} \lambda(x) = \lambda(x_0)$, where E is any subset of R and $x_0 \in E$ .Thus, there exists $z_0 \in R$ such that $T(z_0) = y$ and $\lambda(z_0) = \sup_{T(z)=y} \lambda(z) \geq r$. Hence $y \in T(\lambda_r)$ and therefore $(T(\lambda))_r \subseteq T(\lambda_r)$. Thus we have shown that $(T(\lambda))_r = T(\lambda_r)$. Now if λ is a primary fuzzy ideal of R, it follows by Characterization Theorem 3.3 that $\lambda_r = \begin{cases} \pi.................................r > m \\ R................................r \leq m \end{cases}$, where π is a primary ideal of R and $m = \sup\{i \in [0,1] \mid \lambda_i = R\}$. Thus $(T(\lambda))_r = T(\lambda_r) = \begin{cases} T(\pi)............................r > m \\ R^{/}................................r \leq m \end{cases}$. Since $\lambda_* = \pi \supseteq K$, by assumption both π and $\sqrt{\pi}$ are subtractive. Thus it follows from **(c.f.[12], Theorem 3.21)** that $T(\pi)$ is primary and conversely, if $T(\pi)$ is primary then π is primary. Hence repeal to the characterization Theorem 3.3 yields the required result.

Definition 3.5. Let R be a semiring. A fuzzy ideal λ of R is said to be a maximal fuzzy ideal of R if

(i) λ is not constant.

(ii) For any fuzzy ideal μ of R, if $\lambda \subseteq \mu$ then either $\lambda_* = \mu_*$ or $\mu_* = \chi_R$, where

$\lambda_* = \{x \in R \mid \lambda(x) = \lambda(0)\}$, $\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$

Theorem 3.6. Let R be a semiring and P a nonconstant fuzzy ideal of R. Then P is a maximal fuzzy ideal of R if and only if there exists a maximal ideal π of R such that

$P_r \in \{\pi, R\}$, for all $r \in [0,1]$.

Proof. Let P be a maximal fuzzy ideal of R. Let I(R) consist of the empty set ϕ together with all the ideals of R. We first note that since P is nonconstant, the decreasing function $\psi : [0,1] \mapsto I(R)$) given by $\psi(r) = P_r$ takes on at least one nonempty level set other than

R. In fact, we show that it takes exactly two values $P_t \neq \phi$ and $P_u = R$ for all $r \in [0,1]$, where $t = P(0) > u = P(1)$. If this is not the case, then there exists $s \in (0,1)$ such that $P_t \subset P_s \subset R$. Define a fuzzy ideal λ of R by $\lambda_r = \begin{cases} P_s \text{..................................} r > t \\ R \text{.................................} r \leq t \end{cases}$. Then $P \subseteq \lambda$ since $P_r \subseteq \lambda_r$ for all $r \in [0,1]$, $P_* = P_t \neq P_s = \lambda_*$ and $\lambda \neq \chi_R$. This contradicts the maximal character of P. Thus there exists an ideal $\pi (\neq R)$ of R such that $P_r \in \{\pi, R\}$, for all $r \in [0,1]$. It now remains to prove that π is maximal ideal of R. Let M be an ideal of R such that $\pi \subset M \subset R$. Define a fuzzy ideal μ of R by $\mu_r = \begin{cases} M \text{.................................} s < r \leq 1 \\ R \text{.................................} 0 \leq r < s \end{cases}$, where $1 > s = \sup\{r \in [0,1] \mid P_r = R\}$. Clearly, $P \subseteq \mu, \mu_* = M \neq \pi = P_*$ and $\mu \neq \chi_R$. This again contradicts the maximal character of P. Thus π is a maximal ideal of R. Conversely assume that $P_r \in \{\pi, R\}$ for all $r \in [0,1]$, where π is a maximal ideal of R. Suppose that μ is a fuzzy ideal of R such that $P \subseteq \mu$. Then $\mu(0) \geq P(0) = 1$, so that $\mu(0) = 1$. It now follows that $\pi = P_* \subseteq \mu_*$, which by virtue of the maximal character of π implies that either $\mu_* = P_*$ or $\mu_* = R$. The later alternative yields that $\mu = \chi_R$, since $\mu(0) = 1$. Hence P is a maximal fuzzy ideal of R. This completes the proof of the theorem.

Lemma 3.7. Let R be a semiring and P a fuzzy ideal of R such that $P_r = \begin{cases} \pi \text{..................................} r > a \\ R \text{.................................} r \leq a \end{cases}$. Then for any positive integer n,

$$\left(P^n\right)_r = \begin{cases} \pi^n \text{..................................} r > a \\ R \text{.................................} r \leq a \end{cases}, \text{ where } P^n = P \circ P \circ P \circ \text{................} \circ P.$$

Proof. This follows easily by induction on n using (**c.f. [10], Lemma 3.1**).

Theorem 3.8. Let R be a Noetherian semiring and P any primary fuzzy ideal of R. Then $\sqrt{P}$ is a prime fuzzy ideal of R.

Proof: Let P be a primary fuzzy ideal of R. Then $P_r = \begin{cases} \pi \text{.......................} r > m \\ R \text{.......................} r \leq m \end{cases}$, where $0 \leq m < 1$ and $\pi (\neq R)$ is a primary ideal of R. Now, π being primary and R being Noetherian, it follows that $\sqrt{\pi}$ is the smallest prime ideal containing π (**c.f. [12], Theorem 3.20(iii)).** Define a fuzzy ideal Q of R by $Q_r = \begin{cases} \sqrt{\pi} \text{.......................} r > m \\ R \text{.......................} r \leq m \end{cases}$. Then

Q is prime (**c.f. [10], Theorem 3.4).** In fact Q is the smallest prime fuzzy ideal containing P and $P_* \subseteq Q_*$ .Hence $\sqrt{P} = Q$ is prime.

3. CHARACTERIZATION OF G-PRIMARY/G-MAXIMAL FUZZY IDEALS OF A SEMIRING

In this section, we characterize G-primary/G-maximal fuzzy ideals of R in terms of G-primary/G-maximal ideals of R and derive the fuzzy analogous of the results proved in **[13].**

Throughout this section, R is a semiring and G a finite group acting on R. For any $g \in G$ the action of $g \in G$ on $r \in R$ is denoted by $r \mapsto r^g$. For any subset $A \subseteq R$ and $g \in G$, $A^g = \{a^g \mid a \in A\}$. Any ideal A of a semiring R is said to be G-invariant if $A^g = A$, where $A^G = \bigcap_{g \in G} A^g$. More generally, A^G is the largest G-invariant ideal contained in A.

Definition 4.1. A G-invariant fuzzy ideal λ is said to be a G-prime fuzzy ideal of R if either $\lambda = \chi_R$ or λ is nonconstant and for any two G-invariant fuzzy ideals μ and τ of R, $\mu \circ \tau \subseteq \lambda$ implies that either $\mu \subseteq \lambda$ or $\tau \subseteq \lambda$.

Lemma 4.2. Let λ and μ be two G-invariant fuzzy ideals of a semiring R. Then

(i) $(\sqrt{\lambda})^G = \cap\{P \mid P \in G(\rho_\lambda)\}$, *where* $\mathrm{G}(\rho_\lambda)$ is the family of all G-prime fuzzy ideals of R such that $\lambda \subseteq \mathrm{P}$ and $\lambda_* \subseteq \mathrm{P}_*$

(ii) $\left((\sqrt{\lambda})\right)_*^G = (\sqrt{\lambda_*})^G$.

(iii) $\left(\sqrt{\lambda}\right)^G(x) = 1$, for all $x \in \left(\left(\sqrt{\lambda}\right)^G\right)_*$.

(iv) $\left(\sqrt{\left(\sqrt{\lambda}\right)^G}\right)^G = \left(\sqrt{\lambda}\right)^G$.

(v) $\lambda \subseteq \left(\sqrt{\mu}\right)^G\}$ if and only if $\lambda \subseteq \sqrt{\mu}$.

Proof. Results (ii) - (v) are easy consequences of result (i) and result (i) follows in a manner analogous to the proof of **(c.f.[13] , Lemma 3.2)** upon using **(c.f. [10], Theorem 3.4).**

In order to define G-primary fuzzy ideal, we first state a Lemma which is a direct consequence of **(c.f. [13], Lemma 3.2(vi))** and **Lemma 4.2 (v).**

Lemma 4.3. Let R be a semiring and G finite group acting on R. Then for any G-invariant fuzzy ideal P, the following conditions are equivalent:

(i) For any two G-invariant fuzzy ideals λ and μ of a semiring R, $\lambda \circ \mu \subseteq P$ implies that either $\lambda \subseteq P$ or $\mu \subseteq \left(\sqrt{P}\right)^G$

(ii) For any two G-invariant fuzzy ideals λ and μ of R, $\lambda \circ \mu \subseteq P$ implies that either $\lambda \subseteq P$ or $\mu \subseteq \left(\sqrt{P}\right)$

Definition 4.4. A G-invariant fuzzy ideal λ is called G-primary if either $\lambda = \chi_R$ or λ is nonconstant fuzzy ideal of R satisfying either of the equivalent conditions of Lemma 5.3.

Obviously, if P is a primary fuzzy ideal of R, then P^G is a G-primary fuzzy ideal of R.

Theorem 4.5. Let R be a semiring and P a nonconstant fuzzy ideal of R. Then P is a G-primary fuzzy ideal of R if and only if $P_r \in \{\pi, R\}$, for all $r \in [0,1]$., where $\pi (\neq R)$ is a G-primary ideal of R.

Proof. This follows exactly making necessary changes in Theorem 3.3 as **(c.f. [10], Theorem 4.3)** follows from **(c.f. [10], Theorem 3.4).**

We now, use the characterization **Theorem 4.5** to derive the fuzzy analogous of various results proved in above section 4. We append these results as

Theorem 4.6. (i) Let $P_1, P_2, P_3, ...P_n$, be G-primary fuzzy ideals of a semiring R such that $\left(\sqrt{P_i}\right)^G = \mu$ (i=1, 2 ,..., n) . Then $P = \bigcap_{i=1}^{n} P_i$ is G-primary and $\left(\sqrt{P}\right)^G = \mu$.

(ii) Let R and R′ be two semirings and $T : R \mapsto R'$ is an onto homomorphism. Let λ be a G- invariant fuzzy ideal of R such that both λ_* and $\sqrt{\lambda_*}$ are subtractive and $K \subseteq \lambda_*$ where $K = \{x \in R \mid x = a + b, T(a) = T(b)\}$. If λ is G-primary, then $T(\lambda))$ is G-primary. Moreover, if range λ is finite and $T(\lambda)$ is G-primary, then λ G-is primary.

(iii) Let R be a Noetherian semiring and P a G-primary fuzzy ideal of R. Then $\left(\sqrt{P}\right)^G$ is a G-prime fuzzy ideal of R.

Proof. A slight change in the proof of **Theorem 3.4((i) and (ii))** and **Theorem 3.8** provide us the proof of the results (i)---(iii).

Definition 4.7. Let λ be a G-invariant fuzzy ideal of R. Then λ is called a G-maximal fuzzy ideal of R if λ nonconstant and for any G-invariant fuzzy ideal μ of R, if $\lambda \subseteq \mu$ then either $\lambda_* \subseteq \mu_*$ or $\mu = \chi_R$.

Finally, making necessary changes in **Theorem 3.6**, we get

Theorem 4.8. Let R be a semiring. Let P be a G-invariant fuzzy ideal of R. Then P is a G-maximal fuzzy ideal of R if and only if there exists a G-maximal ideal π of R such that $P_r \in \{\pi, R\}$, for all $r \in [0,1]$.

Corollary 4.9. Let M be a G-maximal fuzzy ideal of R. Then any positive power of M is G-primary and $\left(\sqrt{M^n}\right)^G = M$.

Proof. Let M be a G-maximal fuzzy ideal of R. Then by characterization of G-maximal fuzzy ideals, we have $M_r = \begin{cases} \pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots r > m \\ R \ldots\ldots\ldots\ldots\ldots\ldots\ldots r \le m \end{cases}$, where π is a G-maximal ideal of R. Thus, by Lemma 3.7 for any positive power n of M,

$\left(M^n\right)_r = \begin{cases} \pi^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r > m \\ R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r \le m \end{cases}$. Now π^n is G-primary and π is the smallest G-prime ideal of R containing π^n **(c.f. [13], Corollary 3.8)**. Hence M^n is G-primary and M is the smallest G-prime fuzzy ideal R containing M^n, that is $\left(\sqrt{M^n}\right)^G = M$.

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Viscoelastic Slip Flow through an Inclined Vertical Channel

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Abstract

An unsteady MHD slip flow of incompressible electrically conducting viscoelastic fluid through an inclined vertical porous channel with thermal diffusion is studied. The plates of the channel are taken to be non-conducting, porous subjected to constant injection/suction velocity. A uniform magnetic field is applied perpendicular to plates of the channel. Taking Hall effect and heat source into account a closed form analytical solutions of the governing equations for velocity, temperature and the concentration fields are obtained. The effects of different parameters entering into the problem are shown graphically.

Keywords: Slip boundary condition, Viscoelastic fluid, Inclined channel, Heat source, Soret effect.

Introduction

The science of magnetohydrodynamics (MHD) was concerned with geophysical and astrophysical problems for a number of years. In recent years, the possible use of MHD is to affect a flow stream of an electrically conducting fluid for the purpose of thermal protection, breaking, propulsion control etc. MHD plays an important role in many engineering and industrial problems such as liquid metal cooling, in nuclear reactors, plasma confinement, control of molten iron flow and many others. It continues to attract the attention of applied mathematical researchers owing to extensive applications in contest of ionized aerodynamics. During the past few decades, there has been a growing interest in non-Newtonian fluids. The flow of non-Newtonian fluids are found in a variety of applications: from drilling oil and gas well and well completion operations to industrial process involving waste fluid, synthetic fibres foodstuffs and extrusion of molten plastic as well as in same flows of some polymer solutions. The large variety of fluids and industrial applications has been a major motivation for research in non-Newtonian flows. Moreover, the flows of non-Newtonian fluid in the absence as well as in the presence of magnetic field have applications in many areas such as the handling of biological fluids, alloys, plasma, mercury amalgams, blood and electromagnetic propulsion. Sarpankaya [11] was the first who has studied the MHD flow of non-Newtonian fluids. Walter [13] has studied the non-Newtonian effects in some elastic-viscous liquid whose behaviour at small rate of shear is characterized by a general linear equation of state. Hayat et al. [5] worked on three dimensional flow over a stretching surface in a viscoelastic fluid. Attia [2] studied the Hall current effect on transient Hydromagnetic Couette-Poiseuillle flow of a viscoelastic fluid with heat transfer. Attia and Ewis [1] investigated unsteady MHD Couette flow with heat transfer of viscoelastic fluid under exponential decaying pressure gradient.

Recently slip condition has become much more compelling and it now reasonably certain that the viscous fluid can slip against solid surfaces if the surface is very smooth. The slip boundary has significant applications in lubrication, extrusion, medical science and many others. Muhammad and Alam [7] have studied the slip effect on fractional viscoelastic fluid. Chand and Kumar [3] studied Hall effect on heat and mass transfer in the flow of oscillating viscoelastic fluid through porous medium with wall slip condition. Kumar and Chand [6] have investigated the effect of slip condition and hall currents on unsteady MHD flow of viscoelastic fluid past an infinite vertical porous plate through porous medium. Recently Kumar et al. [4] worked on oscillatory free convective flow of viscoelastic fluid through porous medium in rotating vertical channel.

The process involving coupled heat and mass transfer occurs frequently in nature. In different geophysical cases, it occurs not only due to the temperature difference but also due to concentration difference or the combination of the two. Also natural, mixed and forced convection in inclined channels has been accumulated in previous works in literature because of its practical applications including electronic system, high performance heat exchangers, chemical process equipments, combustion chambers, environmental control system and so forth.

Motivated by above researches and their applications, in the present problem an attempt has been made to extend the recent study of Manglesh et al. [9] to formulate and analyse the effect of slip condition on MHD flow of viscoelastic fluid through an inclined vertical channel.

Formulation of the Problem

The geometry of the system under consideration is shown schematically in Fig. 1. It consists of an inclined channel whose inclination is $\delta, (0 \leq \delta \leq \frac{\pi}{2})$. A Cartesian co-ordinate system is introduced such that x^*-axis lies vertically upward along the centreline of the channel and y^*-axis is perpendicular to the walls of the channel. The injection/suction velocity and permeability of the porous medium is considered to be constant. The vertical plates of the channel are situated at $y^* = \frac{-d}{2}$and $y^* = \frac{d}{2}$. A constant magnetic field is applied perpendicular to the axis of the channel and the effect of induced magnetic field is neglected, which is a valid assumption on laboratory scale under the assumption of small magnetic Reynolds number. Since the plates are infinite in extent in x^* and z^* directions, so all the physical quantities depend only on y^* and t^*.

The velocity components in x^*, y^* and z^* direction are u^*, v^* and w^* respectively. The equation of continuity $\nabla.\vec{V} = 0$, on integration gives $v^* = v_0$. Also when the strength of magnetic field is strong one cannot neglect the effect of Hall current. The components of electric current density $\vec{J}$ are given by (j_x^*, j_y^*, j_z^*), then equation of conservation of electric charge $\nabla.\vec{J} = 0$, gives $j_y^* =$ constant. Since the plates are electrically non-conducting, $j_y^* = 0$ and is zero everywhere in the flow. When the magnetic field is large, the generalized Ohm's law, in the absence of electric field, neglecting the ion slips and thermo electric effect (Meyer [10]) yields:

$$j_x^* - \omega_e \tau_e j_z^* = -\sigma \mu_e H_0 w^* \tag{1}$$

$$j_z^* + \omega_e \tau_e j_x^* = \sigma \mu_e H_0 u^* \tag{2}$$

The solution of equations (1) and (2) are:

$$j_x^* = \frac{\sigma \mu_e H_0}{1+m^2}(mu^* - w^*) \tag{3}$$

$$j_z^* = \frac{\sigma \mu_e H_0}{1+m^2}(u^* + mw^*) \tag{4}$$

Following Skelland [12] the governing equations of viscoelastic flow are obtained as follow

$$\frac{\partial u^*}{\partial t^*} + v_0 \frac{\partial u^*}{\partial y^*} = \vartheta \frac{\partial^2 u^*}{\partial y^{*2}} - k_0 \frac{\partial^3 u^*}{\partial t^* \partial y^{*2}} - \frac{\sigma \mu_e^2 H_0^2}{\rho(1+m^2)}(u^* + mw^*) + g\beta T^* \cos\delta + g\beta^* C^* \cos\delta - \frac{\vartheta}{K_p^*} u^* \tag{5}$$

$$\frac{\partial w^*}{\partial t^*} + v_0 \frac{\partial w^*}{\partial y^*} = \vartheta \frac{\partial^2 w^*}{\partial y^{*2}} - k_0 \frac{\partial^3 u^*}{\partial t^* \partial y^{*2}} + \frac{\sigma \mu_e^2 H_0^2}{\rho(1+m^2)}(mu^* - w^*) - \frac{\vartheta}{K_p^*} w^* \tag{6}$$

The heat conduction equation is

$$\frac{\partial T^*}{\partial t^*} + v_0 \frac{\partial T^*}{\partial y^*} = \frac{k}{\rho C_p} \frac{\partial^2 T^*}{\partial y^{*2}} + Q_0 \tag{7}$$

The mass diffusion equation is

$$\frac{\partial C^*}{\partial t^*} + v_0 \frac{\partial C^*}{\partial y^*} = D_m \frac{\partial^2 C^*}{\partial y^{*2}} + D_T \frac{\partial^2 T^*}{\partial y^{*2}} \tag{8}$$

where '*' represents dimensional quantities, ϑ is kinematic viscosity, t^* is time, ρ is density, H_0 is intensity of magnetic field, T^* is temperature, C_p is specific heat at constant pressure, k is thermal conductivity, g is acceleration due to gravity, β is volumetric coefficient of thermal expansion, C^* is concentration, β^* is volumetric coefficient of thermal expansion with concentration, D_m is chemical molecular diffusivity, D_T is thermal diffusivity.

The initial boundary conditions are

$$\left.\begin{aligned} u^* = L^*\frac{\partial u^*}{\partial y^*}, w^* = L^*\frac{\partial w^*}{\partial y^*}, T^* = C^* = 0, \text{at } y^* = -\frac{d}{2} \\ u^* = 0 = w^*, T^* = T_0\cos\omega^* t^*, C^* = C_0\cos\omega^* t^* \text{ at } y^* = \frac{d}{2} \end{aligned}\right\} \tag{9}$$

Introducing the following non-dimensional quantities

$$u = \frac{u^*}{v_0}, w = \frac{w^*}{v_0}, y = \frac{y^*}{d}, \theta = \frac{T^*}{T_0}, C = \frac{C^*}{C_0}, t = \frac{t^*\vartheta}{d^2}, \omega = \frac{\omega^* d^2}{\vartheta}, K_p = \frac{K_p^*}{d^2},$$

$$\lambda = \frac{v_0 d}{\vartheta}, G_r = \frac{g\beta T_0 d^2}{v_0\vartheta}, G_m = \frac{g\beta^* C_0 d^2}{v_0\vartheta}, M = \frac{\sigma\mu_e^2 H_0^2 d^2}{\mu}, P_r = \frac{\mu c_p}{k}, S_C = \frac{\vartheta}{D_m}, S_0 = \frac{D_T T_0}{\vartheta C_0}$$

$$h = \frac{L^*}{d}, \alpha = \frac{k_0}{d^2}, S = \frac{Q_0 d^2}{\vartheta}$$

where ω is frequency parameter, λ is suction parameter, G_r is Grashoff number, G_m is modified Grashoff number, M is Hartmann number, P_r is Prandtl number, N is radiation parameter, S_c is Schmidt number, h is slip parameter, and α is viscoelastic parameter, S_0 is Soret number and S is heat source parameter, equations (5)-(8) become

$$\frac{\partial u}{\partial t} + \lambda\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \alpha\frac{\partial^3 u}{\partial t\,\partial y^2} - \frac{M}{1+m^2}(u + mw) + G_r\theta\cos\delta + G_m C\cos\delta - \frac{u}{K_p} \tag{10}$$

$$\frac{\partial w}{\partial t} + \lambda\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} - \alpha\frac{\partial^3 w}{\partial t\,\partial y^2} + \frac{M}{1+m^2}(mu - w) - \frac{w}{K_p} \tag{11}$$

$$\frac{\partial\theta}{\partial t} + \lambda\frac{\partial\theta}{\partial y} = \frac{1}{P_r}\frac{\partial^2\theta}{\partial y^2} + S\theta \tag{12}$$

$$\frac{\partial C}{\partial t} + \lambda\frac{\partial C}{\partial y} = \frac{1}{S_c}\frac{\partial^2 C}{\partial y^2} + S_0\frac{\partial^2\theta}{\partial y^2} \tag{13}$$

The relevant boundary conditions in non-dimensional form are given by

$$\left.\begin{aligned} u = h\frac{\partial u}{\partial y}, w = h\frac{\partial w}{\partial y}, \theta = C = 0 \text{ at } y = -\frac{1}{2} \\ u = w = 0, \theta = \cos\omega t, C = \cos\omega t \text{ at } y = \frac{1}{2} \end{aligned}\right\} \tag{14}$$

Introducing the complex velocity $F = u + iw$, we find that equations (10) and (11) can be combined into a single equation of the form

$$\frac{\partial F}{\partial t} + \lambda\frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial y^2} - \alpha\frac{\partial^3 u}{\partial t\,\partial y^2} - \frac{M}{1+m^2}(1 - im)F + G_r\theta\cos\delta + G_m C\cos\delta - \frac{F}{K_p} \tag{15}$$

The corresponding boundary conditions reduce to

$$\left.\begin{aligned} F = h\frac{\partial F}{\partial y}, \theta = C = 0, \text{at } y = -\frac{1}{2} \\ F = 0, \theta = C = \cos\omega t \text{ at } y = \frac{1}{2} \end{aligned}\right\} \quad (16)$$

Solution of the Problem

In order to solve Eqs. (12), (13) and (15), under the boundary condition (16), we assume the solution of these equations as follows

$$\left.\begin{aligned} F(y,t) = F_0(y)e^{i\omega t} \\ \theta(y,t) = \theta_0(y)e^{i\omega t} \\ C(y,t) = C_0(y)e^{i\omega t} \end{aligned}\right\} \quad (17)$$

Substituting these expressions in equations (12), (13), (15) and (16), we obtain

$$(1 - iA)F_0'' - \lambda F_0' - cF_0 = -G_r\theta_0 \cos\delta - G_m C_0 \cos\delta \quad (18)$$

$$\theta_0'' - \lambda P_r\theta_0' - P_r(i\omega - S)\theta_0 = 0 \quad (19)$$

$$C_0'' - \lambda S_c C_0' - i\omega S_c C_0 = -S_0 S_c \theta_0'' \quad (20)$$

Corresponding boundary condition becomes:

$$\left.\begin{aligned} F_0 = h\frac{\partial F_0}{\partial y}, \theta_0 = C_0 = 0, \text{at } y = -\frac{1}{2} \\ F_0 = 0, \theta_0 = C_0 = 1 \text{ at } y = \frac{1}{2} \end{aligned}\right\} \quad (21)$$

The solution of equation (18), (19) and (20) under boundary condition (21) is

$$F(y,t) = (A_9 e^{r_2 y} + A_{10} e^{s_2 y} - A_5 e^{ry} - A_6 e^{sy} - A_7 e^{r_1 y} - A_8 e^{s_1 y})e^{i\omega t} \quad (22)$$

$$\theta(y,t) = (A_0 e^{ry} + B_0 e^{sy})e^{i\omega t} \quad (23)$$

$$C(y,t) = (A_3 e^{r_1 y} + A_4 e^{s_1 y} + A_1 e^{ry} + A_2 e^{sy})e^{i\omega t} \quad (24)$$

Note: The validity and correctness of the present solution is verified by taking $\alpha = h = \delta = 0$ and $\frac{\partial p}{\partial x} = 0$, the above problem reduces to MHD flow in vertical channel and the result becomes the same as given by Manglesh et al. [8]

The shear stress, Nusselt number and Sherwood number can now be obtained easily from equations (22), (23) and (24).

Skin friction coefficient τ_L at the left plate in terms of its amplitude and phase is:

$$\tau_L = \left(\frac{\partial F}{\partial y}\right)_{y=-\frac{1}{2}} = \left(\frac{\partial F_0}{\partial y}\right)_{y=-\frac{1}{2}} e^{i\omega t} = |D|\cos(\omega t + \alpha_1) \tag{25}$$

with $|D| = \sqrt{D_r^2 + D_i^2}$ and $\alpha_1 = \tan^{-1}\left(\frac{D_i}{D_r}\right)$

where $D_r + iD_i = r_2A_9e^{-\frac{r_2}{2}} + s_2A_{10}e^{-\frac{s_2}{2}} - rA_5e^{-\frac{r}{2}} - sA_6e^{-\frac{s}{2}} - r_1A_7e^{-\frac{r_1}{2}} - s_1A_8e^{-\frac{s_1}{2}}$

Heat transfer coefficient Nu (Nusselt number) at the left plate in terms of its amplitude and phase is:

$$Nu = \left(\frac{\partial \theta}{\partial y}\right)_{y=-\frac{1}{2}} = \left(\frac{\partial \theta_0}{\partial y}\right)_{y=-\frac{1}{2}} e^{i\omega t} = |H|\cos(\omega t + \beta) \tag{26}$$

with $|H| = \sqrt{H_r^2 + H_i^2}$ and $\beta = \tan^{-1}\left(\frac{H_i}{H_r}\right)$

where $H_r + iH_i = rA_0e^{-\frac{r}{2}} + sB_0e^{-\frac{s}{2}}$

Mass transfer coefficient Sh (Sherwood number) at the left plate in term of amplitude and phase is:

$$Sh = \left(\frac{\partial C}{\partial y}\right)_{y=-\frac{1}{2}} = \left(\frac{\partial C_0}{\partial y}\right)_{y=-\frac{1}{2}} e^{i\omega t} = |G|\cos(\omega t + \gamma) \tag{27}$$

with $|G| = \sqrt{G_r^2 + G_i^2}$ and $\gamma = \tan^{-1}\left(\frac{G_i}{G_r}\right)$

where $G_r + iG_i = r_1A_3e^{-\frac{r_1}{2}} + s_1A_4e^{-\frac{s_1}{2}} + rA_1e^{-\frac{r}{2}} + sA_2e^{-\frac{s}{2}}$

Results and Discussions

The problem of MHD free convective flow of viscoelastic fluid under the effect of thermal diffusion and heat source through an inclined porous vertical channel in the presence of Hall current is analysed. The analytical expressions for velocity, temperature and concentration are obtained and evaluated numerically for different values of parameter appeared in the solution. To have better insight of the physical problem the variations of the physical quantities with flow parameters are shown graphically.

Velocity Profile: Variation of velocity with different parameters is shown in Figs. 2 and 3. From these figures it is observed that the velocity distribution is parabolic in nature. Fig.2 shows the variation of velocity with Grashoff number, suction parameter, permeability of porous medium, slip parameter, viscoelastic parameter and inclination angle. We find that Velocity increases with increasing Grashoff number and modified Grashoff number as increase in these two parameters significantly increases the buoyancy forces which resulted into rapid enhancement of fluid velocity and this is shown in Fig.2

and Fig.3. It is clear from Fig.2 that velocity decreases with increase of suction parameter indicating the usual fact that suction stabilize the boundary layer growth. Further it is observed that velocity increases with increasing permeability of porous medium as it act against the porosity of the porous medium. Also with the increase in slip parameter the frictional forces reduces and increase the fluid velocity. From the same figure we observed that with increase in viscoelastic parameter the velocity decreases. With the increase in inclination angle, the magnitude of driving forces decreases, hence velocity decreases.

Fig.3 shows the variation of velocity profile with Hartmann number, Hall current parameter, Prandtl number and heat source parameter. From the figure we find that as Hartmann number increases the resistive type of force called Lorentz force increases which slows down the motion of fluid. Further we find that increasing Hall current parameter reduces the effective conductivity and which in turn reduces the magnetic damping force on velocity and consequently velocity increases. Fig. 3 also depict that by increasing Prandtl number viscosity of the fluid increases which makes the fluid thick and causes a decrease in velocity. Further it is noticed that with the increase in heat source parameter velocity increases.

Temperature profile: The temperature profile is shown in Fig. 4 for various parameters involved in the solution. It is clear from Fig.4 that as Prandtl number increases, the temperature profile decreases. Also temperature profile decreases with increasing frequency of oscillation and suction parameter and shown in the same figure

Concentration profile: Fig. 5 illustrates that fluid concentration increases with an increase in Soret number it is because of the reason that a rise in Soret number causes a greater chemical thermal diffusivity. It is also clear from the same figure that with an increase in Schmidt number concentration decreases. This is attributed to the fact that higher values of S_c amounts to fall in the chemical molecular diffusivity i.e. less diffusion therefore takes place by species transfer causing a reduction in concentration.

Variation of Skin friction with Grashoff number at left plate is shown in Fig.6. From the figure we observed that skin friction coefficient increases with increasing viscoelastic and Hall current parameter whereas it decreases with increasing inclination angle, slip parameter, Hartmann number and Schmidt number. Variation of heat transfer coefficient at left plate with suction parameter is shown in Fig.7. This figure clearly shows that the Nusselt number increases with increasing heat source parameter but it decreases with an increase in Prandtl number and frequency of oscillation. Variation of mass transfer coefficient at left plate with suction parameter is given in Fig.8. The figure depicts that Sherwood number increases with increasing Soret number and decreases with Schmidt number.

Conclusions

1. It is concluded that in an inclined vertical channel velocity decreases as angle of inclination increases.
2. Viscoelastic parameter reduces the velocity of the fluid.
3. Slip parameter and heat source parameter increase the velocity of the fluid.
4. Velocity and temperature both increases with an increase in heat source parameter.

Appendix

$$A = \alpha\omega \quad c = \frac{M}{1+m^2}(1-im) + i\omega + \frac{1}{K_p} \quad r = \frac{\lambda P_r + \sqrt{\lambda^2 P_r^2 - 4SP_r + 4i\omega P_r}}{2}$$

$$s = \frac{\lambda P_r - \sqrt{\lambda^2 P_r^2 - 4SP_r + 4i\omega P_r}}{2} \quad r_1 = \frac{S_c\lambda + \sqrt{S_c^2\lambda^2 + 4i\omega S_c}}{2}$$

$$s_1 = \frac{S_c\lambda - \sqrt{S_c^2\lambda^2 + 4i\omega S_c}}{2} \quad r_2 = \frac{\lambda + \sqrt{\lambda^2 + 4c(1-iA)}}{2(1-iA)} \quad s_2 = \frac{\lambda - \sqrt{\lambda^2 + 4c(1-iA)}}{2(1-iA)}$$

$$A_0 = \frac{-e^{\frac{-s}{2}}}{2\sin h\left(\frac{s-r}{2}\right)} \quad B_0 = \frac{e^{\frac{-r}{2}}}{2\sin h\left(\frac{s-r}{2}\right)} \quad A_1 = \frac{-S_0 S_c A_0 r^2}{r^2 - S_c\lambda r - i\omega S_C} \quad A_2 = \frac{-S_0 S_c B_0 s^2}{s^2 - S_c\lambda s - i\omega S_C}$$

$$A_3 = \frac{-1}{\sin h\left(\frac{s_1-r_1}{2}\right)}\left\{\frac{e^{\frac{-s_1}{2}}}{2} + A_1 \sin h\left(\frac{s_1-r}{2}\right) + A_2 \sin h\left(\frac{s_1-s}{2}\right)\right\} \quad A_5 = \frac{(G_r A_0 + G_m A_1)\cos\delta}{(1-iA)r^2 - \lambda r - c}$$

$$A_4 = \frac{1}{\sin h\left(\frac{s_1-r_1}{2}\right)}\left\{\frac{e^{\frac{-r_1}{2}}}{2} + A_1 \sin h\left(\frac{r_1-r}{2}\right) + A_2 \sin h\left(\frac{r_1-s}{2}\right)\right\} \quad A_6 = \frac{(G_r B_0 + G_m A_2)\,co}{(1-iA)s^2 - \lambda s - c}$$

$$A_7 = \frac{G_m A_3 \cos\delta}{(1-iA){r_1}^2 - \lambda r_1 - c} \quad A_8 = \frac{G_m A_4 \cos\delta}{(1-iA){s_1}^2 - \lambda s_1 - c}$$

$$A_9 = \frac{1}{\left\{(1-hr_2)e^{\frac{(s_2-r_2)}{2}} - (1-hs_2)e^{\frac{-(s_2-r_2)}{2}}\right\}}\begin{bmatrix} A_5\left\{(1-hr)e^{\frac{(s_2-r)}{2}} - (1-hs_2)e^{\frac{-(s_2-r)}{2}}\right\} \\ +A_6\left\{(1-hs)e^{\frac{(s_2-s)}{2}} - (1-hs_2)e^{\frac{-(s_2-s)}{2}}\right\} \\ +A_7\left\{(1-hr_1)e^{\frac{(s_2-r_1)}{2}} - (1-hs_2)e^{\frac{-(s_2-r_1)}{2}}\right\} \\ +A_8\left\{(1-hs_1)e^{\frac{(s_2-s_1)}{2}} - (1-hs_2)e^{\frac{-(s_2-s_1)}{2}}\right\} \end{bmatrix}$$

$$A_{10} = \frac{-1}{\left\{(1-hr_2)e^{\frac{(s_2-r_2)}{2}} - (1-hs_2)e^{\frac{-(s_2-r_2)}{2}}\right\}} \begin{bmatrix} A_5\left\{(1-hr)e^{\frac{(r_2-r)}{2}} - (1-hr_2)e^{\frac{-(r_2-r)}{2}}\right\} \\ +A_6\left\{(1-hs)e^{\frac{(r_2-s)}{2}} - (1-hr_2)e^{\frac{-(r_2-s)}{2}}\right\} \\ +A_7\left\{(1-hr_1)e^{\frac{(r_2-r_1)}{2}} - (1-hr_2)e^{\frac{-(r_2-r_1)}{2}}\right\} \\ +A_8\left\{(1-hs_1)e^{\frac{(r_2-s_1)}{2}} - (1-hr_2)e^{\frac{-(r_2-s_1)}{2}}\right\} \end{bmatrix}$$

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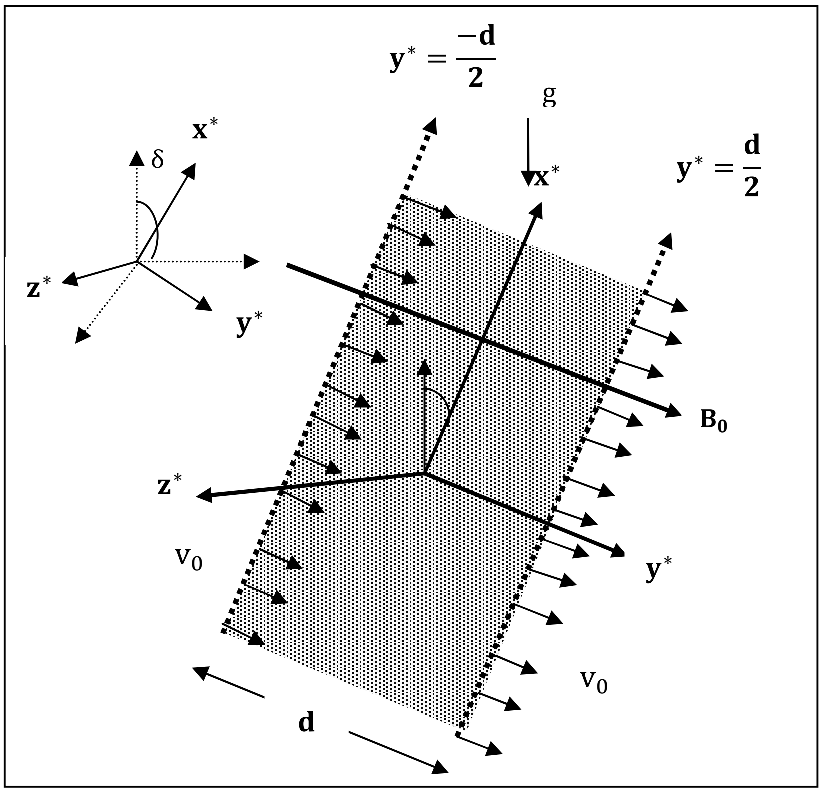

Fig. 1: Schematic presentation of the physical problem.

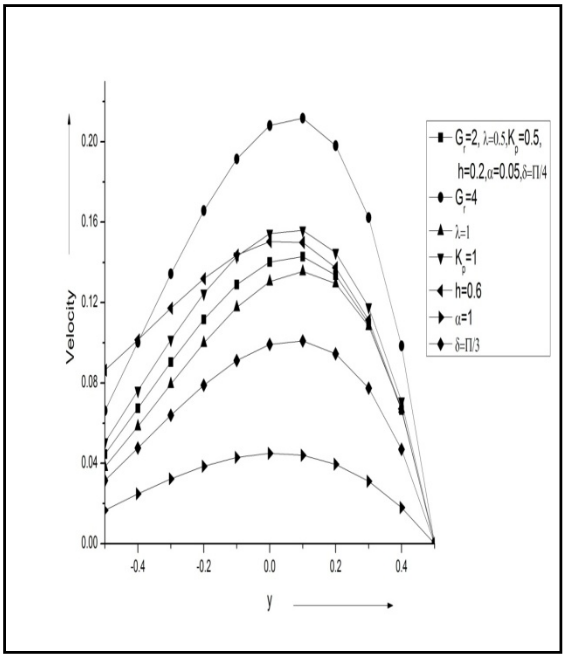

Fig.2 Variation of velocity profile with G_r, λ, K_p, h, α and δ.

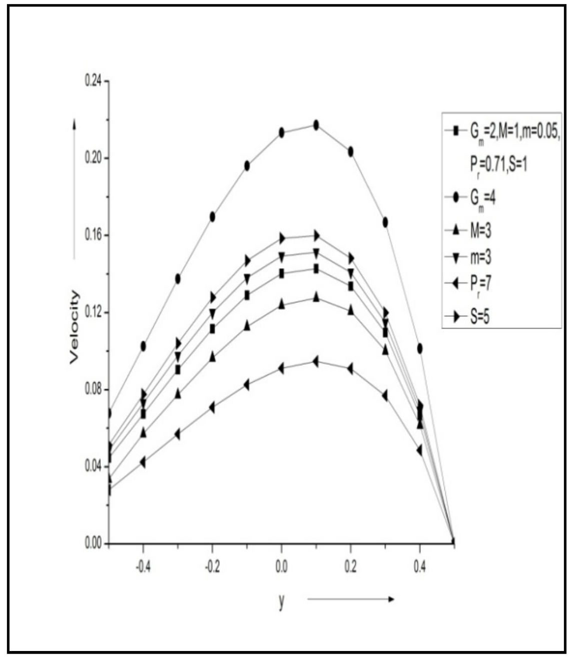

Fig.3 Variation of velocity profile with G_m, M, m, P_r and S.

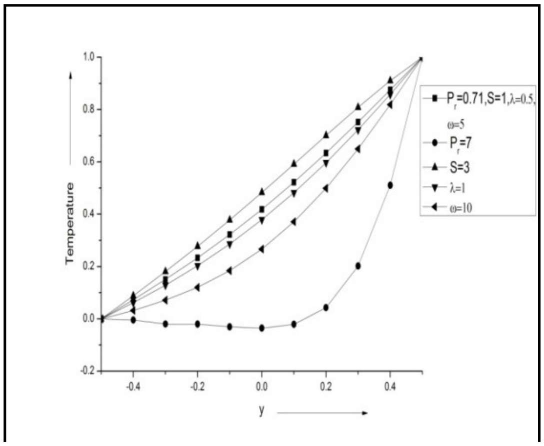

Fig.4 Variation of temperature profile.

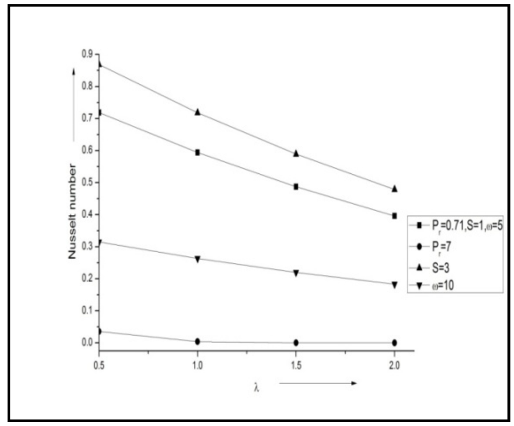

Fig.7 Variation of Nusselt number .

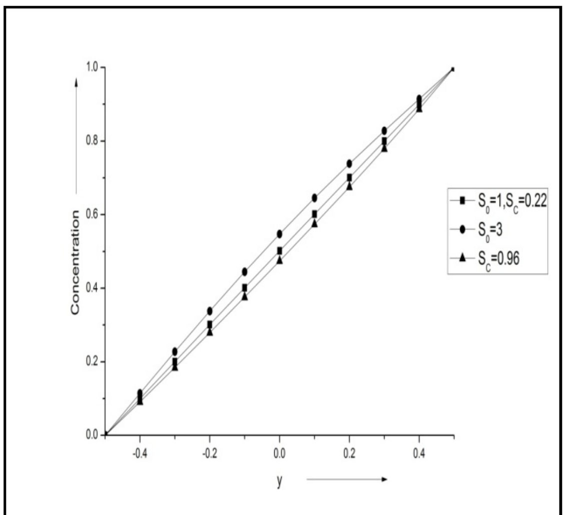

Fig.5 Variation of concentration profile.

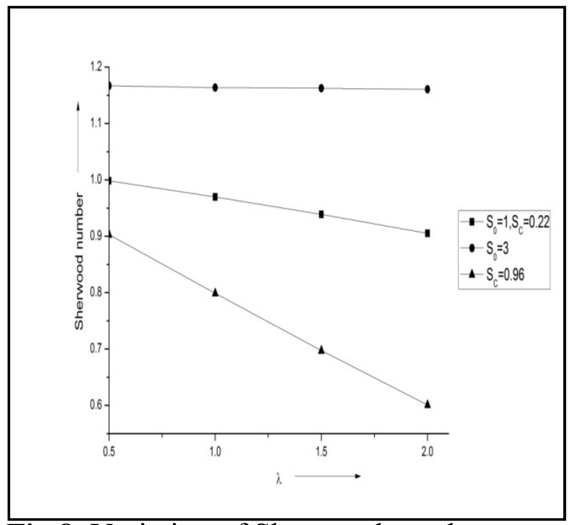

Fig.8 Variation of Sherwood number

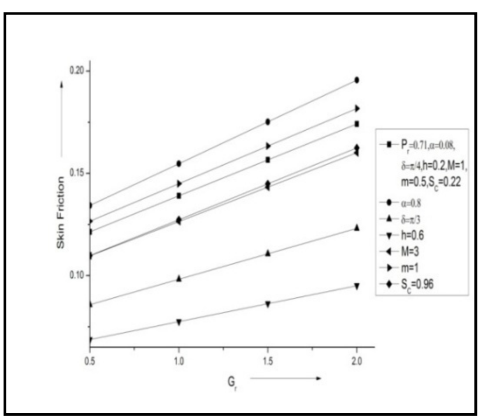

Fig.6 Variation of Skin friction

Effects of Finite Larmor Radius and Suspended Particles on the Magnetogravitational Instability of a Viscoelastic Medium

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Abstract

In the present paper, the effects of finite Larmor radius and suspended particles on the onset of Jeans instability of a self-gravitating viscoelastic medium permeated with uniform magnetic field is studied mathematically using Generalized Hydrodynamic model. A general dispersion relation for the problem is derived using the normal mode analysis method and particular dispersion relations for the transverse and longitudinal modes of wave propagation under both strongly and weakly coupling limits are deduced, which describe the growth rate of instability in terms of various parameters of the problem. The effects of coupling parameter (viscoelasticity), magnetic viscosity (finite Larmor radius), shear viscosity and number density of particle on the growth rate of the gravitational instability are studied numerically and the obtained results are depicted graphically. It is found that the coupling parameter modifies the Jeans Instability criterion, whereas the magnetic viscosity and suspended particles have no effect on this criterion. Further, the coupling parameter, magnetic viscosity and shear viscosity decreases the growth rate and hence have the stabilizing effect on the gravitational instability.

Keywords: Jeans instability; viscoelastic medium; Generalized Hydrodynamic model; magnetic field; finite Larmor radius; suspended particles; strongly/weakly coupling limits.

1. Introduction

In astrophysical scenarios, the simplest theory that describes the aggregation of masses in space is the Jeans instability. The system comprises of particles that can aggregate together depending on the relative magnitude of the gravitational force to pressure force. Whenever the internal pressure of a gas is too weak to balance the self-gravitational force of a mass density perturbation, a collapse occurs. Such a mechanism was first studied by Jeans (1929). In terms of the wavelengths of a fluctuation, the Jeans criterion says that instability follows for all perturbations of wave number less than a critical value k_c, where $k_c=\sqrt{\frac{4\pi G}{{c_s}^2}}$,where ρ is the density, c_s is the velocity of sound in the gas and G is the gravitational constant. This criterion is now known as the Jeans criterion of gravitational instability. Chandrasekhar (1961) studied the Jeans instability problem of a self - gravitating homogeneous medium to investigate the effects of rotation and magnetic field on the onset of gravitational instability in a comprehensive manner and concluded that the Jeans criterion remains unaffected by the individual or simultaneous presence of uniform rotation and magnetic field.

In recent years, the researchers have shown keen interest in the matter present in the cosmological objects like white dwarf, interior of heavy planets, atmosphere of neutron star and ultra cold neutral plasma. Studies have shown that these objects are composed of strongly coupled plasma (SCP) or viscoelasticfluid which shows both viscous and elastic behavior. This behavior of viscoelastic fluid have been discussed by Kaw and Sen (1998) using the Generalized Hydrodynamic (GH) model. The GH model describes the effects of strong correlations through the introduction of viscoelastic coefficients. These viscoelastic coefficients are the functions of coupling parameter Γ_j, which characterizes the ratio of the electrostatic Coulomb interaction between neighboring plasma particles to the thermal (kinetic) energy of the particles and is given by $\Gamma_j = \frac{z_j^2 e^2}{aT_j}$; where the subscripts $j(= e, i)$ denotes respectively, electrons and ions; $a = \left({}^{3}/_{4n_j}\right)^{1/3}$ is the average spacing between particles with density n_j and z_j is the charge state of species j. They used the GH model, to study the dynamics of strongly coupled plasma (SCP) and suggested that the viscoelastic properties of the medium are characterized by the relaxation time τ which provides a characteristic timescale to distinguish two classes of low frequency modes; one when the frequency $\sigma \ll 1/\tau$ (known as hydrodynamic limit) and the other for frequencies $\sigma \gg 1/\tau$ (known as kinetic limit); where,σ is the wave frequency and τ is the viscoelastic relaxation time. For more details and recent views on the subject, one may refer to Janaki and Chakrabarti (2010), Banerjee et al (2010 a,b), Rosenberg and Shukla (2011), Janaki et al (2011) and Prajapati and Chhajlani (2013).

Sharma and Chhajlani (2013) reported that the presence of magnetic field in plasma introduces some additional scales both spatially and temporarily, such as Larmor radius and Larmor frequency. In this connection, the MHD set of equations is derived from two fluid theories with electron and ions with some limitations to describe low frequency phenomena. For the description of plasma along with two fluid theories, the single MHD set of equations is also used to describe the magnetized plasma system. These MHD works are derived by considering the zero Larmor radius of electron and ion and the frequency is generally assumed much smaller than the electron-ion gyration frequency. However, there are a number of interesting situations both in laboratory, space and astrophysical plasmas where the above spatial and temporal ordering does not hold. Sharma and Chhajlani (2013) also reported that in this type of situations, the behavior of considered plasma system is described by assuming finite Larmor radius (FLR) of ion and the correction in this regard is called as FLR corrections.

The effect of FLR on the Jeans instability of self-gravitating classical plasma is reported by many investigators. Rosenbluth et al (1962) and Roberts and Taylor (1962) have pointed out the importance of finite ion Larmor radius effects on the various plasma instabilities. They showed that FLR effect works as a type of viscosity called magnetic viscosity, in which the Larmor radius takes the place of the usual mean free path. Hans (1966) has studied the gravitational instability with Hall current and FLR effects. Bhatia (1969) has discussed the Jeans instability, including Larmor radius corrections and collisional effects. Herrnegger (1972) has studied the effects of collision and gyro viscosity on gravitational instability in two-component plasma. Bhatia and Gupta (1973) have investigated the FLR effects on the gravitational instability of composite plasma. Sharma (1974) has studied the gravitational instability of rotating plasma with FLR effects. Bhatia and Chhonkar (1985) investigated the stabilizing effect of FLR on the instability of rotating layer of self-gravitating plasma. Thus, using MHD set of equations with modification of FLR, various instability problems have been investigated by many authors using magnetic viscosity. Sharma and Chhajlani (2013) studied the effect of finite Larmor radius corrections on the Jeans instability of quantum plasma and concluded that in the transverse mode of wave propagation the instability criterion gets modified due to the presence of both FLR and quantum corrections.

The problem of fluid dynamics by considering the effect of suspended particles on the onset of Bénard convection, gravitational and magneto gravitational instabilities of an infinite homogeneous medium has been studied many authors including Scanlon and Segel (1973). They studied the effects of suspended particles on the onset of Bénard convection and concluded that the particles decrease the critical temperature difference for the onset of convection by increasing the heat capacity of the fluid. In astrophysical context, the problem of gravitational instability of a gas in the presence of suspended particles is more realistic and important. Sharma (1975) studied the effect of suspended particle on the Jeans instability and concluded that Jean's criterion is a sufficient condition for the instability of an infinite, homogeneous magnetized self-gravitational gas particle medium. Raghavachar (1979) and Chhajlani and Sanghvi (1985) also studied the problem in the presence of suspended particle and rotation and found that the Jeans criterion determines the instability criterion. Chhajlani and vyas (1988) studied the effect of thermal conductivity on the gravitational instability of magnetized rotating plasma through a porous medium in the presence of suspended particles and found that the concentration of the suspended particles reduces the rotational effect. Pensia et al. (2012) also studied the role of *Coriolis* force and suspended particle in the fragmentation of matter in the central region of galaxy.

Motivated by the above discussed importance of the finite Larmor radius corrections and suspended particle in certain astrophysical situations and the effect on the Jeans instability problems, we in the present paper have studied the onset of gravitational instability of a self-gravitating infinitely electrically conducting strongly coupled plasma permeated with uniform magnetic field in the presence of finite Larmor radius and suspended particles. Our objective here is to study the effects of finite Larmor radius and suspended particles on the onset of gravitational instability in a self-gravitating strongly coupled viscoelastic medium, mathematically. The present paper thus extends the analysis of Dhiman and Sharma (2014) to include the effects of finite Larmor radius corrections and suspended particles on the gravitational instability of a viscoelastic medium.

2. Mathematical Formulation of the Problem

Consider an infinite homogeneous, finitely electrically conducting, self-gravitating viscoelastic fluid permeated with a uniform magnetic field $\overrightarrow{H_0} = (0, 0, H_z)$ in the presence of suspended particles and finite Larmor radius. We have used the GH model to describe the viscoelastic properties of the medium. Following the analysis of Janaki et al (2011), under these assumptions, the generalized basic hydrodynamical equations of continuity, motion, magnetic induction and Poisson equation governing this physical problem (cf. Janaki et al 2011, and Prajapati and Chhajlani 2013) are given by;

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{u}) = 0 \quad (1)$$

$$\left(1 + \tau \frac{\partial}{\partial t}\right)\left[\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right) - \frac{1}{4\pi}\left((\nabla \times \overrightarrow{H_0}) \times \overrightarrow{H_0}\right) - \rho \nabla \phi + c_s^2 \nabla \rho - KN(\vec{u} - \vec{v}) + \nabla \overleftrightarrow{P}\right] = \mu \nabla^2 \vec{v} + \left(\xi + \frac{\mu}{3}\right)\nabla(\nabla.\vec{v}) \quad (2)$$

$$\frac{\partial \overrightarrow{H_0}}{\partial t} = \nabla \times \left(\vec{v} \times \overrightarrow{H_0}\right) \quad (3)$$

$$\nabla.\overrightarrow{H_0} = 0 \quad (4)$$

$$\nabla^2 \phi = -4\pi G \rho \quad (5)$$

$$\rho_s \frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u} = KN(\vec{v} - \vec{u}) \quad (6)$$

$$\frac{\partial N}{\partial t} = -\nabla.(N\vec{u}) \quad (7)$$

In the above equations;$\vec{v}$, $\vec{u}$, $\overrightarrow{H_0}, \vec{r}$ and $\overleftrightarrow{P}$ are respectively represents the medium velocity, the particle velocity, magnetic field, position vectors and pressure tensor; τ, ρ, K, N, ρ_s and ϕ, respectively denotes the viscoelastic relaxation time, density of fluid, the constant in the Stokes drag formula, the number density of particle, suspended particle density and gravitational potential;μ, G $and c_s$ respectively denote the coefficient of viscosity, the universal gravitational constant and the speed of sound in isothermal medium. Further,$\xi (= \lambda + \frac{2}{3}\mu)$ is coefficient of bulk viscosity.

The components of pressure tensor, considering the finite ion gyration radius, as given by Roberts and Taylor (1962) with magnetic field, in the z direction are given by

$$P_{xx} = -\rho v_0 \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right),\ P_{yy} = \rho v_0 \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right), P_{zz} = 0, P_{xy} = P_{yx} = \rho v_0 \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right), P_{xz} = P_{zx} = -2\rho v_0 \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right), P_{yz} = P_{zy} = 2\rho v_0 \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \quad (8)$$

$v_0 = \frac{\Omega_L R_L^2}{4}$ is the magnetic viscosity. Here R_L is the ion Larmor radius and Ω_L is the ion gyration frequency.

3. Linearized perturbation equations and dispersion relation

To investigate the instability of the self-gravitating system governed by basic equations (1)-(8), let the initial stationary (steady) solution be slightly perturbed by giving infinitesimal small perturbations $\delta\rho$, $\delta\phi$, $\vec{h}(h_x, h_y, h_z)$, $\delta\overleftrightarrow{P}$, $\vec{V}(v_x, v_y, v_z)$ and $\vec{U}(u_x, u_y, u_z)$ in the density ρ_0, gravitational potential ϕ_0, magnetic field $\overrightarrow{H_0}$, pressure tensor $\overleftrightarrow{P}$, fluid velocity $\vec{v}$ and particle velocity $\vec{u}$ respectively.

Thus, the perturbations are represented by

$\rho = \rho_0 + \delta\rho,\ \phi = \phi_0 + \delta\phi,\ \overleftrightarrow{P} = \overleftrightarrow{P_0} + \delta\overleftrightarrow{P},\ \vec{H} = \overrightarrow{H_0} + \vec{h}, (0 + h_x, 0 + h_y, H_z + h_z),\ \vec{v} = \vec{V}(0 + v_x, 0 + v_y, 0 + v_z),\ \vec{u} = \vec{U}(0 + u_x, 0 + u_y, 0 + u_z).$ (9)

Using these perturbed quantities given in (9) in equations (1)-(8) and then linearizing the resulting equations by neglecting the second and higher order perturbed quantities, we get the following linearized perturbed equations of continuity, motion, magnetic induction and Poisson equation for viscoelastic medium respectively;

$$\frac{\partial \delta\rho}{\partial t} + \nabla.(\rho_0 \vec{V}) = 0 \quad (10)$$

$$\left(1 + \tau\frac{\partial}{\partial t}\right)\left[\rho_0 \frac{\partial \vec{V}}{\partial t} - \rho_0 \nabla\delta\phi - \frac{1}{4\pi}(\nabla \times \vec{h}) \times \overrightarrow{H_0} + c_s^2 \nabla\delta\rho + \nabla.\delta\overleftrightarrow{P} - KN(\vec{V} - \vec{U})\right] = \mu\nabla^2\vec{V} + \left(\xi + \frac{\mu}{3}\right)\nabla(\nabla.\vec{V}) \quad (11)$$

$$\frac{\partial \vec{h}}{\partial t} = \nabla \times (\vec{V} \times \vec{H}_0) \quad (12)$$

$$\nabla^2\delta\phi = -4\pi G\delta\rho \quad (13)$$

$$\left(\tau_1 \frac{\partial}{\partial t} + 1\right)\vec{U} = \vec{V} \quad (14)$$

$$\nabla.\vec{h} = 0 \quad (15)$$

To solve equations (10)-(15), which are linear and homogeneous equations, let us assume the solution of the perturbed quantities of the form;

$$e^{\iota(k_x x + k_z z) + \sigma t} \quad (16)$$

where, σ is the wave frequency and k_x, k_z are the wave numbers in transverse and longitudinal directions. Using this exponential solution in equations (10)-(15) and simplifying the resulting equations, we get the following equations in the velocity components as;

$$(1 + \tau\sigma)\left\{\left(\sigma^2 + {c_s}^2 k_x^2 + k^2 V_a^2 - \frac{w_j^2 k_x^2}{k^2} - A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1 + 1}\right)\right)v_x + \sigma\upsilon_0({k_x}^2 + 2{k_z}^2)v_y\right\} + \sigma\left(\frac{\left(\xi + \frac{4\mu}{3}\right)k_x^2}{\rho_0} + \nu k_z^2\right)v_x = 0 \quad (17)$$

$$(1 + \tau\sigma)\left\{\left(\sigma^2 - {k_z}^2 V_a^2 - A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1 + 1}\right)\right)v_y - 2\sigma\upsilon_0 k_x k_z v_z - \sigma\upsilon_0({k_x}^2 + 2{k_z}^2)v_x\right\} + \sigma\nu k^2 v_y = 0 \quad (18)$$

$$(1 + \tau\sigma)\left(\left\{\sigma^2 + {c_s}^2 {k_z}^2 - \frac{w_j^2 k_z^2}{k^2} - A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1 + 1}\right)\right\}v_z + 2\sigma\upsilon_0 k_x k_z v_y\right) + \sigma\left\{\frac{\left(\xi + \frac{4\mu}{3}\right)k_z^2}{\rho_0} + \nu k_z^2\right\}v_z = 0 \quad (19)$$

Equations (17)-(19) can be put in the following matrix notations;

$$[B][C] = 0 \quad (20)$$

where, [B] is the coefficient matrix and [C] is the velocity components matrix.

The necessary condition for the non-trivial solution of system (20) is that the determinant of the coefficient matrix [B]must vanish, which yields the following characteristic equation

$$(1+\tau\sigma)\left(\sigma^2+{c_s}^2k_x^2+k^2V_a^2-\frac{w_j^2k_x^2}{k^2}-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\sigma\left(\frac{\left(\xi+\frac{4\mu}{3}\right)k_x^2}{\rho_0}+\nu k_z^2\right)\left[\left\{(1+\tau\sigma)\left(\sigma^2-{k_z}^2V_a^2-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\sigma\nu k^2\right\}\left\{(1+\tau\sigma)\left(\sigma^2+{c_s}^2{k_z}^2-\frac{w_j^2k_z^2}{k^2}-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\sigma\left(\frac{\left(\xi+\frac{4\mu}{3}\right)k_z^2}{\rho_0}+\nu k_z^2\right)\right\}+(1+\tau\sigma)^2 4\sigma^2{\upsilon_0}^2{k_x}^2{k_z}^2\right]+(1+\tau\sigma)^2\sigma^2{\upsilon_0}^2\left({k_x}^2+2{k_z}^2\right)^2\left((1+\tau\sigma)\left\{\sigma^2+{c_s}^2{k_z}^2-\frac{w_j^2k_z^2}{k^2}-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+\sigma\left\{\frac{\left(\xi+\frac{4\mu}{3}\right)k_z^2}{\rho_0}+\nu k_z^2\right\}\right)=0 \quad (21)$$

where, $\omega_j=\sqrt{4\pi G\rho_0}$ is the Jeans frequency, $k^2=k_x^2+k_z^2$ is the wave number and $V_a^2=\frac{H_z^2}{4\pi\rho_0}$ is the Alfvén velocity.

4. Jeans Criterion of Instability

We shall now obtain the dispersion relations for each transverse and longitudinal modes of wave propagation from the characteristic equation (21) and investigate the instability criterion for the onset of gravitational instability in viscoelastic fluid for the strongly and weakly coupled plasmas, individually.

4a. Transverse mode of wave propagation

In the case of transverse mode of wave propagation, let us take $k_x=k$and $k_z=0$. In view of this, equation (21) yields the following dispersion relation

$$\left\{(1+\tau\sigma)\left(\sigma-A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\nu k^2\right\}\times\left[\left\{(1+\tau\sigma)\left(\sigma^2+{c_s}^2k^2+k^2V_a^2-w_j^2-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\sigma\frac{\left(\xi+\frac{4\mu}{3}\right)k^2}{\rho_0}\right\}\times\left\{(1+\tau\sigma)\left(\sigma-A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\nu k^2\right\}+(1+\tau\sigma)^2\sigma{\upsilon_0}^2k^4\right]=0 \quad (22)$$

Further, equation (22) above clearly yields the following pair of equations

$$(1+\tau\sigma)\left(\sigma-A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\nu k^2=0 \quad (23)$$

$$(1+\tau\sigma)^2\left\{\sigma^3+{c_s}^2k^2\sigma+k^2V_a^2\sigma-w_j^2\sigma-2\sigma^2A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)+A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\left\{-{c_s}^2k^2+k^2V_a^2+w_j^2+A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+\sigma{\upsilon_0}^2k^4\right\}+(1+\tau\sigma)\left[\frac{\sigma\left(\xi+\frac{4\mu}{3}\right)k^2}{\rho_0}\left(\sigma-A\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right)+\left\{\sigma^2+{c_s}^2k^2+k^2V_a^2-w_j^2-A\sigma\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}\nu k^2\right]+\frac{\sigma\left(\xi+\frac{4\mu}{3}\right)k^2}{\rho_0}\nu k^2=0 \quad (24)$$

Now, we shall derive the instability criteria for both SCP and WCP.

(i). Criterion for strongly coupled plasma (SCP)

For SCP case, in view of kinetic limit $(\sigma\tau \gg 1)$, equation (23) yields the non-gravitating mode. Also under this limit, equation (24) reduces to the following dispersion relation

$$\sigma^6 + \sigma^5\left(\frac{2}{\tau_1} - 2A\right) + \sigma^4\left\{\frac{1}{{\tau_1}^2} - \frac{2A}{\tau_1} + {c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2 + \frac{\nu k^2}{\tau} + {\upsilon_0}^2 k^4 + A^2\right\} +$$
$$\sigma^3\left\{\frac{2}{\tau_1}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2 + \frac{\nu k^2}{\tau} + {\upsilon_0}^2 k^4\right\} + A\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2 + \frac{\nu k^2}{\tau}\right\}\right\} +$$
$$\sigma^2\left\{\frac{1}{{\tau_1}^2}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2 + \frac{\nu k^2}{\tau} + {\upsilon_0}^2 k^4\right\} + \frac{A}{\tau_1}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2\right\} +\right.$$
$$\left.\frac{\nu k^2}{\tau}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + \frac{A}{\tau_1} + \frac{k^2 \nu_c^2}{\tau_1}\right\}\right\} + \sigma\frac{\nu k^2}{\tau_1 \tau}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2\right\} +$$
$$\frac{\nu k^2}{\tau_1 \tau}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^2 \nu_c^2\right\} = 0 \qquad (25)$$

The constant term of (25) yields the following instability criterion;

$$k^2 < \omega_j^2/({c_s}^2 + \nu_c^2 + V_a^2) \qquad (26)$$

(ii). Criterion for weakly coupled plasma (WCP)

For WCP case, in view of hydrodynamic limit$(\sigma\tau \ll 1)$, equation (24) reduces to the following dispersion relation

$$\sigma^5 + \sigma^4\left\{\frac{2}{\tau_1} - 2A + \nu k^2\right\} + \sigma^3\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^4 \nu_0^2 + \frac{1}{{\tau_1}^2} - \frac{2A}{\tau_1} + \nu k^2\left\{\frac{1}{\tau_1 \rho_0}\left(\xi +\right.\right.\right.$$
$$\left.\left.\left.\frac{4\mu}{3}\right) + \frac{2}{\tau_1} + \frac{k^2}{\rho_0}\left(\xi + \frac{4\mu}{3}\right)\right\}\right\} + \sigma^2\left\{\frac{2}{\tau_1}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + k^4 \nu_0^2\right\} + A\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2\right\} + \frac{A}{\tau_1} +\right.$$
$$\left.\frac{k^2}{\rho_0}\left(\xi + \frac{4\mu}{3}\right) - A + \nu k^2\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + \frac{1}{{\tau_1}^2} - \frac{A}{\tau_1} + \frac{2k^2}{\rho_0}\left(\xi + \frac{4\mu}{3}\right)\right\}\right\} +$$
$$\sigma\left\{\frac{\nu k^2}{\tau_1}\left\{2\left({c_s}^2 k^2 + k^2 V_a^2 - w_j^2\right) + \frac{k^2}{\rho_0}\left(\xi + \frac{4\mu}{3}\right)\right\} + \frac{A}{\tau_1}\left\{-{c_s}^2 k^2 + k^2 V_a^2 + w_j^2\right\} + \frac{1}{{\tau_1}^2}\left\{{c_s}^2 k^2 + k^2 V_a^2 -\right.\right.$$
$$\left.\left. w_j^2 + k^4 \nu_0^2\right\}\right\} + \frac{\nu k^2}{{\tau_1}^2}\left\{{c_s}^2 k^2 + k^2 V_a^2 - w_j^2\right\} = 0 \qquad (27)$$

The constant term of (27) yields the following instability criterion;

$$k^2 < \omega_j^2/({c_s}^2 + V_a^2) \qquad (28)$$

If the effects of suspended particle and viscoelastic fluid are ignored, then the dispersion relation (24) reduces to

$$\sigma^2 + {c_s}^2 k^2 + k^2 V_a^2 - w_j^2 + {\upsilon_0}^2 k^4 = 0 \qquad (29)$$

and the constant term of (29) yields the following instability criterion;

$$k^2 < \omega_j^2/({c_s}^2 + V_a^2 + {\upsilon_0}^2 k^2) \qquad (30)$$

Thus, from inequality (28) and (30) it is clear that the finite Larmor radius modifies the Jeans criterion of instability, if the medium is non viscous and the effect of suspended particles is ignored.

4b. Longitudinal mode of wave propagation

In the case longitudinal mode; let k_z=k and k_x = 0. In view of this equation (21) yields the following dispersion relation

$$\left[(1+\tau\sigma)\left\{\sigma^2+{c_s}^2\mathrm{k}^2-w_j^2-\sigma\mathrm{A}\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+\frac{k^2\sigma}{\rho_0}\left(\xi+\frac{4\mu}{3}\right)\right]\times\left[(1+\tau\sigma)\left\{\sigma^2+\mathrm{k}^2V_a^2-\sigma\mathrm{A}\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+k^2\nu\sigma\right]^2+4(1+\tau\sigma)^2\sigma^2{\upsilon_0}^2\mathrm{k}^4=0 \quad (31)$$

From (31) it is clear that either

$$(1+\tau\sigma)\left\{\sigma^2+{c_s}^2\mathrm{k}^2-w_j^2-\sigma\mathrm{A}\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+\frac{k^2\sigma}{\rho_0}\left(\xi+\frac{4\mu}{3}\right)=0 \quad (32)$$

$$\left[(1+\tau\sigma)\left\{\sigma^2+\mathrm{k}^2V_a^2-\sigma\mathrm{A}\left(\frac{\sigma\tau_1}{\sigma\tau_1+1}\right)\right\}+k^2\nu\sigma\right]^2+4(1+\tau\sigma)^2\sigma^2{\upsilon_0}^2\mathrm{k}^4=0 \quad (33)$$

Now, we shall derive the instability criteria for both SCP and WCP.

(i). Criterion for strongly coupled plasma (SCP)

For SCP case, in view of kinetic limit($\sigma\tau \gg 1$), equation (32) reduces to the following equation

$$\sigma^3+\sigma^2\left\{\frac{1}{\tau_1}-A\right\}+\sigma\left\{{c_s}^2k^2-w_j^2+k^2\nu_c^2\right\}+\frac{1}{\tau_1}\left\{{c_s}^2k^2-w_j^2+k^2\nu_c^2\right\}=0 \quad (34)$$

The constant term of (34) yields the following instability criterion;

$$k^2<\omega_j^2/({c_s}^2+\nu_c^2) \quad (35)$$

(ii). Criterion for weakly coupled plasma (WCP)

For WCP case, in view of hydrodynamic limit($\sigma\tau \ll 1$), equation (32) reduces to the following dispersion relation

$$\sigma^3+\sigma^2\left\{\frac{1}{\tau_1}-A+\frac{k^2}{\rho_0}\left(\xi+\frac{4\mu}{3}\right)\right\}+\sigma\left\{{c_s}^2k^2-w_j^2+\frac{k^2}{\tau_1\rho_0}\left(\xi+\frac{4\mu}{3}\right)\right\}+\frac{1}{\tau_1}\left\{{c_s}^2k^2-w_j^2\right\}=0 \quad (36)$$

The constant term of (36) yields the following instability criterion;

$$k^2<\omega_j^2/{c_s}^2 \quad (37)$$

5. Growth rate of instability

We shall now analyze the effects of various physical parameters viz; viscoelastic parameter, magnetic viscosity, number density of particle and shear viscosity on the growth rate of magneto-gravitational instability of viscoelastic medium for the case of transverse and longitudinal mode of propagation under both the strongly and weakly coupling limits.

In order to study the effect of growth rate on SCP and WCP, writing equations (25), (27), (34) and (36) respectively in the following dimensionless forms;

$$\gamma^6 + 2\gamma^5 \left\{\frac{1}{\tau_1^*} - A^*\right\} + \gamma^4 \left(\frac{1}{\tau_1^{*2}} - \frac{2A^*}{\tau_1^*} + C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2} + \frac{\nu^* k^{*2}}{\tau^*} + k^{*2}\nu_0{}^{*4} + A^{*2}\right) + \gamma^3 \left\{\frac{2}{\tau_1^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2} + \frac{\nu^* k^{*2}}{\tau^*} + k^{*2}\nu_0{}^{*4}\right\} + A^*\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2} + \frac{\nu^* k^{*2}}{\tau^*}\right\}\right\} + \gamma^2 \left\{\frac{1}{\tau_1^{*2}}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2} + \frac{\nu^* k^{*2}}{\tau^*} + k^{*2}\nu_0{}^{*4}\right\} + \frac{A^*}{\tau_1^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2}\right\} + \frac{\nu^* k^{*2}}{\tau^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \frac{A^*}{\tau_1^*} + \frac{\xi^* k^{*2}}{\tau_1^*}\right\}\right\} + \gamma \frac{\nu^* k^{*2}}{\tau_1^* \tau^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2}\right\} + \frac{\nu^* k^{*2}}{\tau_1^* \tau^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + \xi^* k^{*2}\right\} = 0 \tag{38}$$

$$\gamma^5 + \gamma^4 \left\{\frac{2}{\tau_1^*} - 2A^* + \nu^* k^{*2}\right\} + \gamma^3 \left(\frac{1}{\tau_1^{*2}} - \frac{2A^*}{\tau_1^*} + C_s^{*2}k^{*2} + k^{*2} - 1 + k^{*2}\nu_0{}^{*4}\right) + \nu^* k^{*2}\left\{\frac{1}{\tau_1^*}\varsigma + \frac{2}{\tau_1^*} + k^{*2}\varsigma\right\} + \gamma^2 \left\{\frac{2}{\tau_1^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + k^{*2}\nu_0{}^{*4}\right\} + A^*\left\{C_s^{*2}k^{*2} + k^{*2} - 1\right\} + \frac{A^*}{\tau_1^*} - A^* + k^{*2}\varsigma + \nu^* k^{*2}\left\{\frac{1}{\tau_1^{*2}} - \frac{A^*}{\tau_1^*} + C_s^{*2}k^{*2} + k^{*2} - 1 + 2k^{*2}\varsigma\right\}\right\} + \gamma \left\{\frac{\nu^* k^{*2}}{\tau_1^*}\left\{2\left(C_s^{*2}k^{*2} + k^{*2} - 1\right) + k^{*2}\varsigma\right\} + \frac{A^*}{\tau_1^*}\left\{C_s^{*2}k^{*2} + k^{*2} - 1\right\} + \frac{1}{\tau_1^{*2}}\left\{C_s^{*2}k^{*2} + k^{*2} - 1 + k^{*2}\nu_0{}^{*4}\right\}\right\} + \frac{\nu^* k^{*2}}{\tau_1^{*2}}\left\{C_s^{*2}k^{*2} + k^{*2} - 1\right\} = 0 \tag{39}$$

$$\gamma^3 + \gamma^2 \left(\frac{1}{\tau_1{}^*} + A^*\right) + \gamma\left(k^{*2} + k^{*2}\xi^* - 1\right) + \frac{k^{*2}}{\tau_1{}^*}(1 + \xi^*) - \frac{1}{\tau_1{}^*} = 0 \tag{40}$$

$$\gamma^3 + \gamma^2 \left(\frac{1}{\tau_1{}^*} - A^{*2} + k^{*2}\varsigma\right) + \gamma\left(k^{*2} + \frac{k^{*2}\varsigma}{\tau_1{}^*} - 1\right) + \frac{(k^{*2} - 1)}{\tau_1{}^*} = 0 \tag{41}$$

where the dimensionless parameters used are

$$k^* = \frac{k\mathrm{V_a}}{\omega_j}, \upsilon^* = \frac{\upsilon\omega_j}{{V_a}^2}, \xi^* = \frac{\nu_c^2}{{V_a}^2}, C_s^* = \frac{C_s}{V_a}, \nu_0{}^{*2} = \frac{\nu_0^2}{V_a}$$

$$\gamma = \frac{\sigma}{\omega_j}, k^* = \frac{kc_s}{\omega_j}, \upsilon^* = \frac{\upsilon\omega_j}{{c_s}^2}, \xi^* = \frac{\nu_c^2}{{c_s}^2}, A^* = \frac{A}{\omega_j}, \varsigma = \frac{\omega_j}{{c_s}^2\rho_0}\left(\xi + \frac{4}{3}\mu\right), \tau^* = \tau\omega_{j,}\tau_1{}^* = \tau_1\omega_j$$

The values of growth rate of magnetogravitational instability for different values of wave numbers have been calculated from equations (38), (39), (40) and (41) in the transverse and longitudinal mode of wave propagation under the strongly and weakly coupled plasma limits. The obtained values and the variation in the growth rate with wave numbers is depicted graphically in Figures 1and 2, respectively.

Further the effect of magnetic viscosity on the growth rate of magnetogravitational instability has been observed from the non-dimensional equation (38).The obtained values and the variation in the growth rate with wave numbers is depicted graphically in Figure 3 for some constant values of magnetic viscosity $\nu_0{}^* = 0.5, 1.5$.

Figure 1. Variation of normalized growth rate against the normalized wave number (k^*) under the strongly and weakly coupling limits in the longitudinal mode of wave propagation.

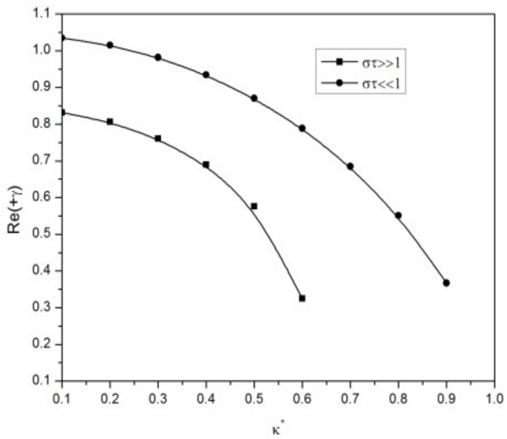


Figure 2. Variation of normalized growth rate against the normalized wave number (k^*) under the strongly and weakly coupling limits in the transverse mode of wave propagation.

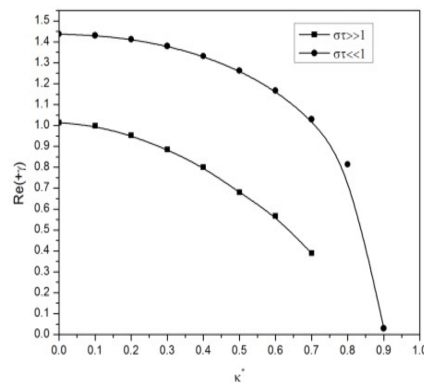


Figure 3. Variation of normalized growth rate against the normalized wave number (k^*) under the strongly coupling limit in the transverse mode of wave propagation for some fixed values of magnetic viscosity $\nu_0^* = 0.5, 1.5$.

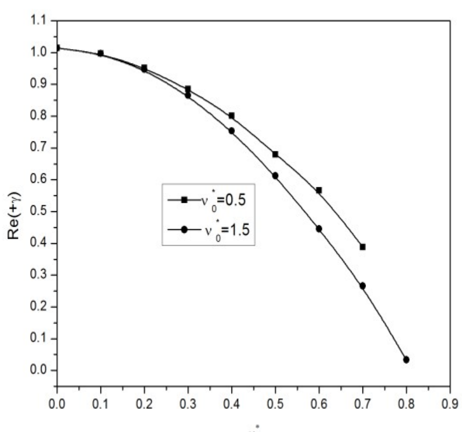


Figure 4. Variation of normalized growth rate against the normalized wave number (k^*) under the strongly in the transverse mode of wave propagation for some fixed values of number density of particle $A^* = 0.0, 0.3$.

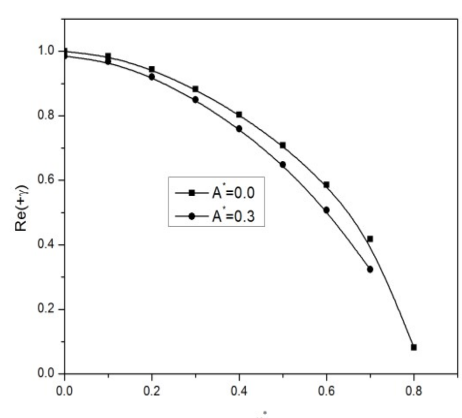


Figure 5. Variation of normalized growth rate against the normalized wave number (k^{*}) under weakly coupling limits in the transverse mode of wave propagation for some fixed values of number density of particle$A^{*} = 0.0, 0.3$

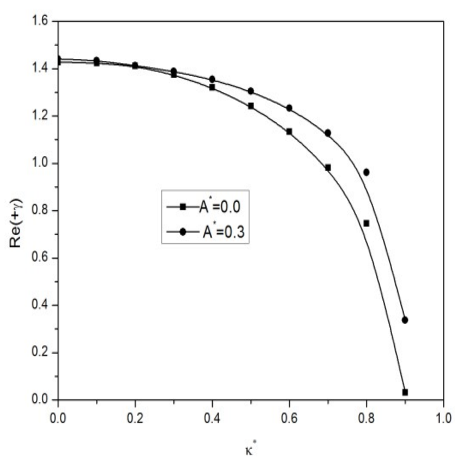
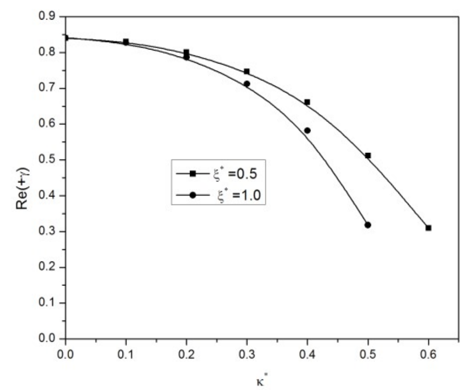


Figure 6. Variation of normalized growth rate against the normalized wave number (k^{*})underthe strongly coupling limits in the longitudinal mode of wave propagation for various values of shear viscosityξ^{*}=0.5,1.0.



The effect of number density of particles A^{*} has been calculated in the transverse mode of wave propagation under the strongly and weakly coupling limits from the non-dimensional equations (38) and (39). The different values of the number density of particle A^{*}=0, 0.3 have been chosen to investigate the effect on the growth rate of magnetogravitational instability. The obtained values and the variation in the growth rate with wave numbers are depicted graphically in Figures 4 and 5, respectively for the strongly and weakly plasma. The shear viscosity effect on the growth rate of magnetogravitational instability has been studied in the longitudinal mode of wave propagation under the strongly coupling limit for the different values of shear viscosity;$\xi^{*} = 0.5, 1.0$ from the non-dimensional equation (40). The obtained values and the variation in the growth rate with wave numbers for the different values of shear viscosity $\xi^{*} = 0.5, 1.0$ are depicted graphically in Figure 6.

6. Results and Discussions

In the present paper, we have studied the effects of finite Larmor radius and suspended particles on the onset of gravitational instability of a self-gravitating viscoelastic medium permeated with uniform magnetic field, mathematically using Generalized Hydrodynamic model. A general dispersion relation for the problem is derived using the normal mode analysis method and particular dispersion relations for the transverse and longitudinal modes of wave propagation under both strongly and weakly coupling limits are obtained, which describe the growth rate of instability in terms of various parameters of the problem. The effects of finite Larmor radius and suspended particles have been investigated on both the longitudinal and transverse mode of wave propagation under the strongly and weakly coupling limits.

Form the above analysis, we found that the coupling parameter modifies the Jeans instability criterion, whereas the magnetic viscosity and suspended particles have no effect on this criterion. The effects of coupling parameter (viscoelasticity), magnetic viscosity (finite Larmor radius), shear viscosity and number density of particle on the growth rate of the gravitational instability are studied numerically and the results are depicted graphically. The variation of the growth rate under both the strongly and weakly coupled limits (coupling parameter) with normalized wave number has been calculated and the results have been depicted graphically in Figures 1 and 2. From the obtained results, it is observed that the growth rate is higher in the weakly coupled plasma to that of the strongly coupled plasma in both the transverse and longitudinal modes of wave propagation. It may be due to the fact that the decay of growth rate of unstable Jeans modes is faster in the case of strongly rather than the weakly coupling limits (Sharma 2014). Also, the effect of magnetic viscosity on the growth rate of magnetogravitational instability under the strongly coupling limit in the transverse mode of wave propagation for some fixed values of magnetic viscosity $\nu_0^*(= 0.5, 1.5)$ has been investigated and the variation has been depicted in Figure 3. It is observed that as the values of magnetic viscosity increases the growth rate of instability decreases and hence has a stabilizing effect on the growth rate of gravitational instability. The effect of number density of particle for some fixed values of $A^*(= 0.0, 0.3)$ in the transverse mode of wave propagation has been studied and the results are depicted in Figures 4 and 5, respectively under the strongly and weakly coupling limits. It is observed that for the increasing values of the number density of particle, the growth rate decreases and hence have stabilizing effect on the gravitational instability. In Figure 6, the variation of normalized growth rate against the normalized wave number (k^*) under the strongly coupling limit in the longitudinal mode of wave propagation for various values of shear viscosity ξ^* (=0.5, 1.0) has been depicted. It is observed that the shear viscosity have same effect on the growth rate as that of the number density of the particle.

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